THE WEAK DIAMOND

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The weak diamond is the following statement: for every function $F : 2^{<\omega_1} \to 2$, there exists a function $g : \omega_1 \to 2$ such that for every function $f : \omega_1 \to 2$, there are stationarily many $\alpha < \omega_1$ such that $F(f \restriction \alpha) = g(\alpha)$. The following formulation is more intuitive at least to me. for every sequence $\langle F_\alpha : \alpha < \omega_1 \rangle$ of functions with $F_\alpha : \mathcal{P}(\alpha) \to 2$, there exists a function $g : \omega_1 \to 2$ such that for every subset X of ω_1, there are stationarily many $\alpha < \omega_1$ such that $F_\alpha(X \cap \alpha) = g(\alpha)$.

Devlin and Shelah proved the following theorem in [1].

Theorem 0.1. *The weak diamond is equivalent to* $2^{\aleph_0} < 2^{\aleph_1}$.

In particular, CH implies the weak diamond. The purpose of this note is to give a proof to this theorem.

It is easy to see that the weak diamond implies $2^{\aleph_0} < 2^{\aleph_1}$. So, we shall focus on the other direction, i.e. $2^{\aleph_0} < 2^{\aleph_1}$ implies the weak diamond. Suppose not, i.e. $2^{\aleph_0} < 2^{\aleph_1}$ but the weak diamond does not hold. It means that there exists a function $F : 2^{<\omega_1} \to 2$ such that for every function $g : \omega_1 \to 2$, there exists a function $f : \omega_1 \to 2$ such that for club many $\alpha < \omega_1$, $F(f \restriction \alpha) \neq g(\alpha)$. By considering $g' : \omega_1 \to 2$ defined by $g'(\alpha) = 1 - g(\alpha)$, we can see that for every function $g : \omega_1 \to 2$, there exists a function $f : \omega_1 \to 2$ such that for club many $\alpha < \omega_1$, $F(f \restriction \alpha) = g(\alpha)$. This is why this 2-color weak diamond is so distinct from the weak diamond of 3-color or more.

Before going into the details, we will explain our strategy. Let X be the set of all sequences $\langle s_\alpha : \alpha < \omega^2 \rangle$ such that there exists a $\delta < \omega_1$ such that for every $\alpha < \omega^2$, s_α is a function from δ into 2. Note $|X| = 2^{\aleph_0}$. We shall define an injection function $\varphi : 2^{\omega_1} \to X$. Of course, this is a contradiction. To show that φ is injective, we shall define a function $\sigma : X \to 2^{\omega_1}$ such that for every $f \in 2^{\omega_1}$, $\sigma \circ \varphi(f) = f$.

The definition of φ goes as follows. Let $f : \omega_1 \to 2$. Inductively, we shall define a sequence $\langle f_\alpha : \alpha < \omega^2 \rangle$ in 2^{ω_1} with $f_n = f$ for all $n < \omega$. This sequence is designed so that the lower part $\langle f_\alpha \restriction \delta : \alpha < \omega^2 \rangle$ reflect the information about the higher part. φ is defined to be $\langle f_\alpha \restriction \delta : \alpha < \omega^2 \rangle$ for some nice $\delta < \omega_1$. It will be showed that we can reconstruct $\langle f_\alpha : \alpha < \omega^2 \rangle$ from this sequence of short functions. In a sense, we "slide down" the information about one tall function f into a wide sequence of shorter functions.

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Let τ be a bijection from 2^ω onto the set of all countable sequences $\langle s_\alpha : \alpha < \eta\rangle$ such that there exists a $\delta < \omega_1$ such that for every $\alpha < \eta$, s_α is a function from δ into 2.

We begin with the definition of φ. Let $f : \omega_1 \to 2$. We shall define functions f_α and g_α for For every $n < \omega$, let $f_n = f$. For every $\alpha < \omega_1$ and $n < \omega$, let $g_n(\alpha) = F(f \restriction \alpha)$. Define $D_0 = D_1 = \omega_1$. Now suppose that for some $n \in (0, \omega)$, we have defined D_n and f_α and g_α for all $\alpha < \omega n$. For every $\delta < \omega_1$, we shall define $g_\alpha(\delta)$ as follows. If $\delta \notin D_n$, then let $g_\alpha(\delta) = 0$ (this is an ignorable case). Suppose $\delta \in D_n$. Set $\gamma_n = \min(D_n \setminus (\delta + 1))$. Let $x_{n,\delta} \in 2^\omega$ be so that $\tau(x_{n,\delta}) = \langle f_\alpha \restriction \gamma_n : \alpha < \omega n\rangle$. Define $g_{\omega n+m}(\delta) = x_{n,\delta}(m)$.

By assumption, for every $m < \omega$, there exist a $f_{\omega n+m} : \omega_1 \to 2$ such that for club many $\xi < \omega_1$, $F(f_{\omega n+m} \restriction \xi) = g_{\omega n+m}(\xi)$. Let D_{n+1} be a club subset of ω_1 such that for every $\xi \in D_{n+1}$ and $m < \omega$, $F(f_{\omega n+m} \restriction \xi) = g_{\omega n+m}(\xi)$. This completes the definition of f_α and g_α for $\alpha < \omega^2$ and D_n for $n < \omega$. Let $\delta = \min \bigcap_{n<\omega} D_n$ and $\varphi(f) = \langle f_\alpha \restriction \delta : \alpha < \omega^2\rangle$.

The point of this construction is:

(i) Let $n \in (0, \omega)$. For every $m < \omega$ and $\delta \in D_{n+1}$, we have $F(f_{\omega n+m} \restriction \delta) = g_{\omega n+m}(\delta)$. So, if we know $f_{\omega n+m} \restriction \delta$, then we can compute $g_{\omega n+m}(\delta)$.

(ii) If we know $g_{\omega n+m}(\delta)$, of course we can compute $x_{n,\delta}$. Then, we can find $\langle f_\alpha \restriction \gamma_n : \alpha < \omega n\rangle$ where $\gamma_n = \min(D_n \setminus (\delta + 1))$.

(iii) By doing this for every $n \in (0, \omega)$, we can compute $\langle f_\alpha \restriction \delta' : \alpha < \omega n\rangle$ where $\delta' = \min(\bigcap_{n<\omega} D_n \setminus (\delta + 1))$.

Let us do it more formally. We shall define $\bar{\sigma} : X \to X$ as follows. Let $\langle s_\alpha : \alpha < \omega^2\rangle \in X$ and $\mathrm{dom}(s_0) = \delta$ (note that by the definition of X, $\mathrm{dom}(s_\alpha) = \delta$ for every $\alpha < \omega^2$). For every $n \in (0, \omega)$, define $y_{n,\delta} : \omega \to 2$ by for every $m < \omega$, $y_{n,\delta}(m) = F(s_{\omega n+m})$. Let $\langle t_{n,\alpha} : \alpha < \eta_n\rangle = \tau(y_{n,\delta})$. If for every $n \in (0, \omega)$, $\eta_n = \omega n$ and for every $\alpha < \omega n$, $t_{n+1,\alpha}$ is an extension of $t_{n,\alpha}$, then for every $\bar{n} < \omega$ and $\alpha \in [\omega\bar{n}, \omega(\bar{n} + 1))$, let $t_\alpha = \bigcup_{\bar{n}<n<\omega} t_{n,\alpha}$. It is easy to see that $\langle t_\alpha : \alpha < \omega^2\rangle \in X$. Let $\bar{\sigma}(\langle s_\alpha : \alpha < \omega^2\rangle) = \langle t_\alpha : \alpha < \omega^2\rangle$. Otherwise, let $\bar{\sigma}(\langle s_\alpha : \alpha < \omega^2\rangle) = \varnothing$ (this is ignorable).

We shall define σ as follows. Let $\langle s_\alpha : \alpha < \omega^2\rangle \in X$. For each $\alpha < \omega^2$, set $s^0_\alpha = s_\alpha$. We shall inductively define $\langle s^\xi_\alpha : \alpha < \omega^2\rangle \in X$ for all $\xi < \omega_1$. Suppose that $\langle s^\xi_\alpha : \alpha < \omega^2\rangle$ has been defined. Let $\langle t_\alpha : \alpha < \omega^2\rangle = \bar{\sigma}(\langle s^\xi_\alpha : \alpha < \omega^2\rangle)$. If for every $\alpha < \omega^2$, t_α extends s^ξ_α, then we let $s^{\xi+1}_\alpha = t_\alpha$ for every $\alpha < \omega^2$. Otherwise, stop the induction and let $\sigma(\langle s_\alpha : \alpha < \omega^2\rangle)$ be just any function from ω_1 into 2. If ξ is limit, for every $\alpha < \omega^2$, let $s^\xi_\alpha = \bigcup_{\zeta<\xi} s^\zeta_\alpha$. Let $\sigma(\langle s_\alpha : \alpha < \omega^2\rangle) = \bigcup_{\xi<\omega_1} s^\xi_0$.

Now, it suffices to show that for every $f : \omega_1 \to 2$, $\sigma \circ \varphi(f) = f$. Let f_α, g_α, $x_{n,\delta}$, D_n be as in the definition of $\varphi(f)$. Define $D = \bigcap_{n<\omega} D_n$ and let $\langle \delta_\xi : \xi < \omega_1\rangle$ be the increasing enumeration of D. Then, $\varphi(f) = \langle f_\alpha \restriction \delta_0 : \alpha < \omega^2\rangle$.

Claim 1. Let $\delta \in D$. Then, $\bar{\sigma}(\langle f_\alpha \restriction \delta : \alpha < \omega^2 \rangle) = \langle f_\alpha \restriction \delta' : \alpha < \omega^2 \rangle$ where $\delta' = \min(D \setminus (\delta + 1))$.

$\vdash$ For every $n \in (0, \omega)$, define $y_{n,\delta} \in 2^\omega$ by $y_{n,\delta}(m) = F(f_{\omega n+m} \restriction \delta)$. Since $\delta \in D \subseteq D_{n+1}$, we have $g_{\omega n+m}(\delta) = F(f_{\omega n+m} \restriction \delta)$ for every $m < \omega$. Since $\delta \in D \subseteq D_n$, we have $x_{n,\delta}(m) = g_{\omega n+m}(\delta)$. Therefore, we have $x_{n,\delta} = y_{n,\delta}$. So, $\tau(y_{n,\delta}) = \tau(x_{n,\delta}) = \langle f_\alpha \restriction \gamma_n : \alpha < \omega n \rangle$ where $\gamma_n = \min(D_n \setminus (\delta + 1))$. Note that $\sup_{n<\omega} \gamma_n = \delta'$. By the definition of $\bar{\sigma}$, we have $\bar{\sigma}(\langle f_\alpha \restriction \delta : \alpha < \omega^2 \rangle) = \langle f_\alpha \restriction \delta' : \alpha < \omega^2 \rangle$. $\dashv$ (Claim 1)

Let $s_\alpha = f_\alpha \restriction \delta_0$ for every $\alpha < \omega^2$ and define $\langle s^\xi_\alpha : \xi < \omega_1 \text{ and } \alpha < \omega^2 \rangle$ as in the definition of $\sigma(\langle s_\alpha : \alpha < \omega^2 \rangle$.

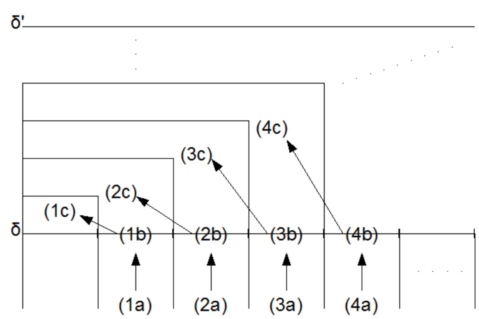
Claim 2. For every $\xi < \omega_1$ and $\alpha < \omega^2$, $s^\xi_\alpha = f_\alpha \restriction \delta_\xi$.

$\vdash$ Go by induction on $\xi < \omega_1$. The case $\xi = 0$ is just by definition. Suppose that $s^\xi_\alpha = f_\alpha \restriction \delta_\xi$ for all $\alpha < \omega^2$. Then, by Claim 1, $s^{\xi+1}_\alpha = f_\alpha \restriction \delta_{\xi+1}$. Suppose that ξ is a limit ordinal and $s^\zeta_\alpha = f_\alpha \restriction \delta_\zeta$ for every $\zeta < \xi$ and $\alpha < \omega^2$. Then,

$$s^\xi_\alpha = \bigcup_{\zeta<\xi} s^\zeta_\alpha = \bigcup_{\zeta<\xi} f_\alpha \restriction \delta_\zeta = f_\alpha \restriction \delta_\xi$$

$\dashv$ (Claim 2)

I do not believe it helps, though I'll put the figure to express my image.



From $\langle f_{\omega n+m} \restriction \delta : m < \omega \rangle$ (shown as (1a), (2a), ...), we can find $\langle g_{\omega n+m}(\delta) : m < \omega \rangle$ and hence $x_{n,\delta}$ (shown as (1b), (2b), ...). Each $x_{n,\delta}$ codes the box $\langle f_\alpha \restriction \gamma_n : \alpha < \omega n \rangle$ where $\gamma_n = \min(D_n \setminus (\delta+1))$ (shown as (1c), (2c), ...). By doing this for all $n \in (0, \omega)$, we can find all $\langle f_\alpha \restriction \delta' : \alpha < \omega^2 \rangle$ where $\delta' = \min(D \setminus (\delta + 1))$. A little surprising thing is that since we only need the values of f_α below δ to find $f_\alpha \restriction \delta'$, we can pass limit stages.

References

1. K. J. Devlin and S. Shelah, *A weak version of $\diamondsuit$ which follows from $2^{\aleph_0} < 2^{\aleph_1}$*, Israel J. Math. **29** (1978), no. 2-3, 239–247.