TORSION ELEMENTS OF THE AUTOHOMEOMORPHISM GROUP OF $\mathbb{R}^2$

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1. Introduction

The main purpose of this paper is to show if f is an autohomeomorphism on $\mathbb{R}^2$ such that $f^n = \mathrm{id}$ for some $n \in \mathbb{Z}^+$ with $n \geq 2E$, then there exists $m \in \mathbb{Z}^+$ such that $m < n$ and f^m has a fixed point. The argument uses covering spaces and braid groups. Necessary definitions and lemmas about them are included without proofs.

2. Covering Space

The definitions and facts about covering spaces used later are listed in this section. See [1] for more information.

Definition 2.1 (Friedl, p.184)**.** Let (X, x_0) be a pointed topological space. We call the set $\pi_1(X, x_0)$ of all path-homotopy classes of loops in (X, x_0) the *fundamental group of X with respect to the base point x_0*. The group operation is defined by concatenation.

Definition 2.2. Let G be a group that acts continuously on a topological space X. We say that G acts *discretely* if and only if for each $x \in X$, there exists an open neighborhood U of x such that $U \cap g \cdot U = \varnothing$ for all $g \neq e$.

Definition 2.3 (Friedl, p.213)**.** Let $p : X \to B$ be a continuous surjection.

(i) The open set U of B is said to be *uniformly covered* if and only if $p^{\leftarrow}U$ is the union of disjoint open subset $\langle V_i \mid i \in I \rangle$ of X such that for each $i \in I$, the map $p \restriction V_i : V_i \to U$ is a homeomorphism.

(ii) We say that the map $p : X \to B$ is a *covering* if and only if p is surjective and if for every $b \in B$, there exists an open neighborhood U of b which is uniformly covered.

(iii) We call X a *covering space of B*.

Remark. In Munkres, 'uniformly covered' is called 'evenly covered'.

Remark. In Friedl, $\sharp A$ means the cardinality of A. We shall convert to $|A|$ here.

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Lemma 2.4 (Covering-Preimage Lemma, Friedl Lemma 14.4)**.** *Let $p : X \to B$ be a covering of topological spaces. If B is a connected topological space, then for any two points P and Q in B, we have*

$$|p^{\leftarrow}\{P\}| = |p^{\leftarrow}\{Q\}|$$

Remark. In Friedl, it is written as $|p^{-1}(P)|$ instead of $|p^{\leftarrow}\{P\}|$.

Definition 2.5. Let $p : X \to B$ be a covering of a connected non-empty topological space.

(i) We define the *degree* $[X : B]$ *of* p as the cardinality of $p^{\leftarrow}(b)$ for some $b \in B$. (Note: By Lemma 2.4, this is well-defined.)

(ii) If the degree is finite, we say that p is a *finite covering.* Otherwise, we say that p is an *infinite covering.* When the degree is finite and $n = [X : B]$, we say that p is an n*-fold covering.*

Remark. Friedl is a little illogical here. It says 'finite covering int the sense of (ii)', but 'finite covering' is not defined in (ii).

Proposition 2.6 (Action-Covering Proposition, Proposition 14.5 in Friedl)**.** *Let X be a topological space together with a discrete and continuous action by a group G.*

(i) The natural projection $p : X \to X/G$ is a covering with the degree is given by $|G|$.

(ii) Let H a subgroup of G. The natural projection,

$$p : X/H \to X/G, [x]_H \mapsto [x]_G$$

is a covering of degree $[G : H]$.

Lemma 2.7 (Covering-Basics Lemma, Lemma 14.6 in Friedl)**.** *Let $p : X \to B$ be a covering of topological spaces.*

(i) The map $p : X \to B$ is open.

(ii) For every $b \in B$, the preimage $p^{\leftarrow}\{b\}$ is a discrete subset of X.

(iii) If B is Hausdorff, then so is X.

(iv) The following two statements are equivalent:

(a) X is compact.

(b) p is a finite covering and B is compact.

Lemma 2.8. *Let $p : X \to B$ be a covering. Let $x \in X$, $b = p(x)$ and W an open neighborhood of b. Then, there exists an open neighborhood U of x such that $p^{\rightarrow}U$ is an open neighborhood of b and $p^{\rightarrow}U \subseteq W$.*

Proof. Since p is a covering, there exists an open neighborhood W' of b that is uniformly covered. So, there exists a mutually disjoint indexed set $\langle U_i \mid i \in I\rangle$ of open subsets of X such that $p^{\leftarrow}W' = \bigcup_{i \in I} U_i$ and for each $i \in I$, $p \restriction U_i : U_i \to W'$ is a homeomorphism. Since $b = p(x)$, we have $x \in p^{\leftarrow}W'$, so there exists an $i \in I$ such that $x \in U_i$. Let $U = U_i \cap p^{\leftarrow}W$. Then, U is an open neighborhood of x, and $p^{\rightarrow}U \subseteq p^{\rightarrow}(p^{\leftarrow}W) = W$. By Lemma 2.7, p is an open map. So, $p^{\rightarrow}U$ is an open neighborhood of b. $\square$

Definition 2.9. Let $p : X \to B$ be a covering and let $f : Y \to B$ be a continuous map between topological spaces. A *lift of f to X* is a continuous map $\tilde{f} : Y \to X$ such that $p \circ \tilde{f} = f$.

Proposition 2.10 (Lift Uniqueness Lemma, Lemma 14.7 in Friedl)**.** *Let $p : X \to B$ be a covering and Y be a connected topological space. Let $f : Y \to B$ be a continuous map, and $\tilde{f}$, $\hat{f}$ are two lifts of f to X. If $\tilde{f}$ and $\hat{f}$ agree at some point $y_0 \in Y$, then they agree everywhere.*

Lemma 2.11 (Lift-Extension Lemma, Lemma 14.8 in Friedl)**.** *Let $p : X \to B$ be a covering and Y be a topological space. We suppose that we have a decomposition $Y = C \cup D$ such that C and D are closed subsets. Let $f : Y \to B$ be a continuous map and let $\tilde{f} : C \to X$ be a lift of $f \restriction C$. We suppose that the following statements hold:*

- *(i) The image $f(D)$ is contained in a uniformly covered open subset of B.*
- *(ii) The intersection $C \cap D$ is connected and non-empty.*

Then, there exists a lift $\tilde{f} : Y \to X$ that agrees with the given lift on C.

Lemma 2.12 (Lift-Existence Lemma, Lemma 14.9 in Friedl)**.** *Let $p : X \to B$ be a covering of topological spaces. Given any continuous map $f : [0,1]^m \to B$ and given any $x \in X$ with $p(x) = f(0)$, there exists a unique lift $\tilde{f} : [0,1]^m \to X$ with $\tilde{f}(0) = x$. (Question: what do $f(0)$ and $\tilde{f}(0)$ mean? The domain of them should be $[0,1]^m$. Answer: It is actually the zero vector).*

Definition 2.13. Let $p : X \to B$ be a covering and let $f : [0,1] \to B$ be a path. Let $x \in X$ be a point with $p(x) = f(0)$. A path $\tilde{f} : [0,1] \to X$ is called a lift of f to the starting point x, if $\tilde{f}$ is a lift of f with $\tilde{f}(0) = x$.

Proposition 2.14 (Path-Lifting proposition, Proposition 14.10 in Friedl)**.** *Let $p : X \to B$ be a covering.*

- *(i) Let $f : [0,1] \to B$ be a path. Given any $\tilde{b} \in X$ with $p(\tilde{b}) = f(0)$, there exists a unique lift $\tilde{f} : [0,1] \to X$ of f to the starting point $\tilde{b}$.*
- *(ii) Let $f, g : [0,1] \to B$ be two paths and let $\tilde{f}, \tilde{g} : [0,1] \to X$ be two lifts with the same starting points. If f and g are path-homotopic, then the endpoints of $\tilde{f}$ and $\tilde{g}$ agree and the path $\tilde{f}$ and $\tilde{g}$ are path-homotopic.*

Remark. In Friedl, 'path-homotopic' is defined on p.176.

Proposition 2.15 (Covering-π_1-Monomorphism Proposition)**.** *Let $p : X \to B$ be a covering. Given any $x_0 \in X$, the induced map $p_* : \pi_1(X, x_0) \to \pi_1(B, p(x_0))$ is a monomorphism.*

Lemma 2.16. *Let $p : X \to B$ be a covering. Let $\bar{f} : [0,1] \to X$ be path from a to b such that $a \neq b$ and $p(a) = p(b)$. Then, $p \circ \bar{f}$ is a loop at $p(a)$ in B that is not null-homotopic.*

Proof. It is trivial that $p \circ \bar{f}$ is a loop at $p(a)$. We shall show that it is not null-homotopic. Suppose that it is null-homotopic. Namely, if we let g be the constant path at $p(a)$, then $p \circ \bar{f}$ and g are path-homotopic. Obviously, $\bar{f}$ is a lift of $p \circ \bar{f}$. Define $\bar{g}(t) = a$ for all $t \in [0, 1]$. Then, $\bar{g}$ is a lift of g. Since $p \circ \bar{f}$ and g are path-homotopic, by Proposition 2.14 (ii), $barf$ and $\bar{g}$ are path-homotopic. However, $\bar{f}(0) \neq \bar{f}(1)$ while $\bar{g}(0) = \bar{g}(1) = a$ This is a contradiction. □

Definition 2.17 (p.199 in Friedl)**.** Let $f : X \to Y$ be a continuous function and let $x_0 \in X$. If $\gamma : [0, 1] \to X$ is a loop in the point x_0, then $f \circ \gamma : [0, 1] \to Y$ is a loop in the point $f(x_0)$. It follows from the Homotopy Composition Lemma (Lemma 8.4) that $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ defined by $f_*([\gamma]) = [f \circ \gamma]$ is well-definfed. We call f_* the *induced map*.

Proposition 2.18 (π_1-Functor Proposition, Proposition 13.1(2) in Friedl)**.** *If $f : X \to Y$ is a continuous map and $f(x_0) = y_0$, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a group homomorphism.*

Lemma 2.19 (Homotopy Composition Lemma, Lemma 8.4 in Friedl)**.** *(i)*

Let $f, f' : X \to Y$, let $\alpha : W \to X$ and $\beta : Y \to Z$ be continuous maps between topological spaces. Furthermore, let $A \subseteq X$ be a possibly empty subset. The following conclusion holds:

- *If f is homotopic to f' rel A, then $f \circ \alpha$ is homotopic to $f' \circ \alpha$ rel $\alpha^{\leftarrow} A$.*
- *If f is homotopic to f' rel A, then $\beta \circ f$ is homotopic to $\beta \circ f'$ rel A.*

Note that 'rel A' means that on A, they take the same value.

(ii) Let X, Y, Z be topological spaces and let $A \subseteq X$, and $B \subseteq Y$. We suppose $f, f' : X \to Y$ and $g, g' : Y \to Z$ are continuous maps such that f and f' are homotopic rel A and such that g and g' are homotopic rel B. If $f(A) = f'(A) \subseteq B$, then $g \circ f$ is homotopic to $g' \circ f'$ rel A.

Remark. In this context, a monomorphism can be considered as an injective homomorphism. This term is used to mean an injection in Bourbaki. However, it is defined to be a left-cancellative morphism in the category theory.

3. Configuration spaces

In this section, we shall define configuration spaces and apply them in our situation. We refer to [2] for more information.

Definition 3.1. Let X be a topological space. The *configuration space $\mathcal{F}_n(X)$ of ordered n-tuples of points in X* is defined to be the set of all $\langle x_1, \dots, x_n \rangle \in X^n$ such that $x_i \neq x_j$ for all $i \neq j$. It is given the subspace topology of X^n.

Definition 3.2. Let S_n be the symmetric group. As usual, S_n acts on $\mathcal{F}_n(X)$ by permutation of the coordinates. Let $\mathcal{C}_n(X)$ be the quotient space $\mathcal{F}_n(X)/S_n$.

Definition 3.3. The fundamental group $\pi_1(\mathcal{C}_n(X))$ is called the *braid group of* X *on* n*-strings.*

$\pi_1(\mathcal{C}_n(\mathbb{R}^2))$ is called the *Artin braid group* and written by B_n.

Fact 3.4. For every $n \geq 1$, B_n is torsion-free.

Theorem 3.5. *Let* f *be an autohomeomorphism on* $\mathbb{R}^2$ *such that there exists an* $n \in \mathbb{Z}^+$ *such that* $f^n = \mathrm{id}$*. Then, there exists* $m \in \mathbb{Z}^+$ *such that* $m < n$ *and* f^m *has a fixed point.*

Proof. Define $g : \mathbb{R}^2 \to \mathcal{F}_n(\mathbb{R}^2)$ by $g(x) = \langle f^i(x) \mid i < n\rangle$. We stipulate $f^0(x) = x$. Trivially, g is a continuous injection. Let $p : \mathcal{F}_n(\mathbb{R}^2) \to \mathcal{C}_n(\mathbb{R}^2)$ be the standard surjection.

Claim 1. For every $x \in \mathbb{R}^2$ and $j < n$, $p(g(x)) = p(g(f^j(x)))$.

Proof. We have

$$p(g(x)) = p(\langle f^i(x) \mid i < n\rangle) = \{\, f^i(x_0) \mid i < n \,\}$$

and

$$\begin{aligned} p(g(f^j(x))) &= p(\langle f^i(f^j(x)) \mid i < n\rangle) = p(\langle f^{i+j}(x)\rangle \mid i < n) \\ &= \{\, f^{i+j}(x) \mid i < n \,\} = \{\, f^i(x) \mid i < n \,\} = p(g(x)) \end{aligned}$$

since $f^n = \mathrm{id}$. □

Let $\gamma : [0,1] \to \mathbb{R}^2$ be any path from x_0 to $f(x_0)$.

Claim 2. $p \circ g \circ \gamma$ is a loop at $p(g(x_0))$ that is not null-homotopic.

Proof. Clearly, $g \circ \gamma$ is a path from $g(x_0)$ to $g(f(x_0))$. Since g is injective and $x_0 \neq f(x_0)$, we have $g(x_0) \neq g(f(x_0))$. By Claim 1, $p(g(x_0)) = p(g(f(x_0)))$. So, trivially $p \circ g \circ \gamma$ is a loop at $p(g(x_0))$. By Lemma 2.16, $p \circ g \circ \gamma$ is not null-homotopic. □

Claim 3. $(p \circ g \circ \gamma)^n$ is null-homotopic.

Proof. For each $i < n$, define $\gamma_i = f^i \circ \gamma$. Then, γ_i is a path from $f^i(x_0)$ to $f^{i+1}(x_0)$. By the definition of f, we have $f^n(x_0) = x_0$. So, γ_{n-1} is a path from $f^{n-1}(x_0)$ to x_0. Thus, the concatenation $\gamma_0 * \gamma_1 * \cdots * \gamma_{n-1}$ is a loop at x_0. Since $\mathbb{R}^2$ is simply connected, this loop is null-homotopic.

Thus, $p \circ g \circ (\gamma_0 * \gamma_1 * \cdots * \gamma_{n-1})$ is a loop at $g(x_0)$ that is null-homotopic. Notice

$$p \circ g \circ (\gamma_0 * \gamma_1 * \cdots * \gamma_{n-1}) = (p \circ g \circ \gamma_0) * (p \circ g \circ \gamma_1) * \cdots * (p \circ g \circ \gamma_{n-1})$$

For every $i < n$ and $t \in [0,1]$, we have

$$\begin{aligned}(p \circ g \circ \gamma_i)(t) &= (p \circ g \circ f^i \circ \gamma(t)) \\ &= (p \circ g)(f^i(\gamma(t))) \\ &= (p \circ g)(\gamma(t)) \text{ (by Claim 1)} \\ &= (p \circ g \circ \gamma)(t)\end{aligned}$$

So, $p \circ g \circ \gamma_i = p \circ \gamma \circ \gamma$. Therefore, $(p \circ g \circ \gamma_0) * (p \circ g \circ \gamma_1) * \cdots * (p \circ g \circ \gamma_{n-1}) = (p \circ g \circ \gamma)^n$. □

This is a contradiction to the fact that $\mathcal{C}_n(\mathbb{R}^2) = B_n$ is torsion-free. □

References

[1] Stefan Friedl. Algebraic topology. URL: `https://friedl.app.uni-regensburg.de/papers/1t-total--algebraic-topology-3.5.-final-v2.pdf`.

[2] Christian Kassel and Vladimir Turaev. Braids and braid groups. In *Braid groups*, volume 247 of *Graduate Texts in Mathematics*, pages 1–66. Springer, New York, 2008.