THE PRECIPITOUSNESS OF TAIL CLUB GUESSING IDEALS

TETSUYA ISHIU

Abstract. From a measurable cardinal, we build a model in which the non-stationary ideal on ω_1 is not precipitous, but there is a precipitous tail club guessing ideal on ω_1.

1. Introduction

Club guessing sequences were introduced by S. Shelah in the 1980s, for example in [8]. Since then, they have been proven to be effective tools to show results under ZFC. There are two types of club guessing sequences, tail club guessing sequences and fully club guessing sequences. In this paper, we shall concentrate on tail club guessing sequences.

When $\vec{C}$ is a tail club guessing sequence on κ, then we can define the filter $\mathrm{TCG}(\vec{C})$ on κ associated with $\vec{C}$, which is called the tail club guessing filter. The definition is essentially due to S. Shelah. The tail club guessing ideal simply refers to the dual ideal of the tail club guessing filter. When Γ is a property of ideals, we say that a filter F has Γ if and only if its dual ideal has Γ. When F is a filter, $\breve{F}$ denotes the dual ideal of F.

There are several results about the precipitousness of tail club guessing ideals. In [10], H. Woodin proved that it is consistent relative to the consistency of a Woodin cardinal that NS_{ω_1} is $\aleph_2$-saturated and there exists a tail club guessing ideal $\vec{C}$ on ω_1 such that $\mathrm{NS}_{\omega_1} = \breve{\mathrm{TCG}}(\vec{C})$, in particular, $\mathrm{TCG}(\vec{C})$ is precipitous. M. Foreman, M. Magidor, and S. Shelah in [3] that if we collapse a Woodin cardinal to ω_2 by the Levy collapse, then NS_{ω_1} is precipitous. In [4], the author showed that every tail club guessing ideal on ω_1 is precipitous in this model. In the same paper, it was thus asked if it is consistent that NS_{ω_1} is not precipitous, but there is a precipitous tail club guessing ideal on ω_1. This is the question we shall answer in this paper. In addition, the model is built from a measurable cardinal. Hence, it also shows that the existence of a precipitous tail club guessing ideal is equiconsistent with the existence of a measurable cardinal.

We follow the standard notations in set theory. Lim stands for the class of limit ordinals. For every ordinal α, $\mathrm{cf}(\alpha)$ denotes the cofinality of α. Meanwhile, for every regular cardinal μ, $\mathrm{Cof}(\mu)$ denotes the class of all ordinals α such that $\mathrm{cf}(\alpha) = \mu$. When X and Y are sets of ordinals, we say that X is *almost contained in* Y and write $X \subseteq^* Y$ if and only if there exists a $\zeta < \sup(X)$ such that $X \setminus \zeta \subseteq Y$. When X is a set of ordinals, let $acc(X)$ denote the set of all $\alpha \in X$ such that $\alpha \cap X$ is unbounded in α. Let $\mathrm{nacc}(X) = X \setminus \mathrm{acc}(X)$. An ordinal α is *indecomposable* if and only if for every $\beta < \alpha$, $\beta + \alpha = \alpha$. When F is a filter on κ, we say that a subset X of κ is *F-positive* if and only if $\kappa \setminus X \notin F$. F^+ denotes the set of all F-positive subsets of κ. We automatically assume that $\dot{x}$ is a name for x. When

P is a forcing notion, we shall write $\dot{G}_P$ for the standard P-name for the generic filter.

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2. Tail club guessing ideals

Many notions in this section can be defined on any uncountable regular cardinal. But we shall focus on ω_1, which is required to prove the main theorems.

The following notions were introduced by S. Shelah in [8] though he used different terminology.

Definition 2.1. Let S a stationary subset of $\omega_1 \cap \mathrm{Lim}$. We say that a sequence $\vec{C} = \langle C_\delta \mid \delta \in S \rangle$ is a *tail club guessing sequence on* S if and only if

(i) for every $\delta \in S$, C_δ is an unbounded subset of δ, and
(ii) for every club subset D of ω_1, there exists a $\delta \in S$ such that $C_\delta \subseteq^* D$.

A tail club guessing ideal, which is the main focus of this paper, is defined as follows.

Definition 2.2. Let $\vec{C} = \langle C_\delta \mid \delta \in S \rangle$ be a tail club guessing sequence on a stationary subset S of $\omega_1 \cap \mathrm{Lim}$. We define the *tail club guessing filter* $\mathrm{TCG}(\vec{C})$ *associated with* $\vec{C}$ as the filter on ω_1 generated by the sets of the form $\{ \delta \in S \mid C_\delta \subseteq^* D \}$ for some club subset D of ω_1. A *tail club guessing ideal* is the dual ideal of a tail club guessing filter.

In [8], S. Shelah showed that $\mathrm{TCG}(\vec{C})$ is countably complete and normal for every tail club guessing sequence $\vec{C}$ on $\omega_1 \cap \mathrm{Lim}$.

As it is written in the Introduction, some models in which $\mathrm{TCG}(\vec{C})$ is precipitous for some tail club guessing sequence $\vec{C}$ are already known. However, it was not known whether it is consistent that for some uncountable regular cardinal κ, NS_κ is not precipitous, but there exists a tail club guessing sequence $\vec{C}$ on $\kappa \cap \mathrm{Lim}$ such that $\mathrm{TCG}(\vec{C})$ is precipitous. Moreover, the consistency strength of the precipitous tail club guessing ideal was not known as a Woodin cardinal was required to build any previously known models of precipitous tail club guessing ideals. We shall answer these two questions in this paper.

The following property of tail club guessing sequences plays an important role in the later sections.

Definition 2.3. Let $\vec{C}$ be a tail club guessing sequence on a stationary subset S of $\omega_1 \cap \mathrm{Lim}$. We say that $\vec{C}$ has order type ε if and only if for every $\delta \in S$, $\mathrm{otp}(C_\delta) = \varepsilon$. The set S is denoted as $\mathrm{dom}(\vec{C})$.

Note that not all club guessing sequences have order types. However, for every tail club guessing sequence $\vec{C} = \langle C_\delta \mid \delta \in S \rangle$, either there exists an $X \in \mathrm{TCG}(\vec{C})$ such that for every $\delta \in X$, $\mathrm{otp}(C_\delta) = \delta$ or there exists an $\varepsilon < \omega_1$ such that $\{ \delta \in S \mid \mathrm{otp}(C_\delta) = \varepsilon \}$ is $\mathrm{TCG}(\vec{C})$-positive.

By the following proposition, we may focus on the case where the order type of a tail club guessing sequence is indecomposable.

Proposition 2.4. *Let $\vec{C} = \langle C_\delta \mid \delta \in S\rangle$ be a tail club guessing sequence on a stationary subset S of $\omega_1 \cap \mathrm{Lim}$. Suppose that $\vec{C}$ has order type ε. Let $\varepsilon_1 < \varepsilon$ and ε_2 be ordinals such that $\varepsilon = \varepsilon_1 + \varepsilon_2$. Then, there exists a tail club guessing sequence $\vec{C}' = \langle C'_\delta \mid \delta \in S\rangle$ on S such that $\vec{C}'$ has order type ε_2 and for every club subset D of ω_1 and $\delta \in S$, $C_\delta \subseteq^* D$ if and only if $C'_\delta \subseteq^* D$. In particular,* $\mathrm{TCG}(\vec{C}) = \mathrm{TCG}(\vec{C}')$

Proof. Since $\vec{C}$ has order type ε, for each $\delta \in S$, C_δ is an unbounded subset of δ suuch that $\mathrm{otp}(C_\delta) = \varepsilon$. Since $\varepsilon_1 < \varepsilon$, there exists a $\eta_\delta \in C_\delta$ such that $\mathrm{otp}(C_\delta \cap \eta_\delta) = \varepsilon_1$. So, $\mathrm{otp}(C_\delta \setminus \eta_\delta) = \varepsilon_2$. For each $\delta \in S$, let $C'_\delta = C_\delta \setminus \eta_\delta$. Then, trivially for every club subset D of ω_1 and $\delta \in S$, $C_\delta \subseteq^* D$ if and only if $C'_\delta \subseteq^* D$. □

The following properties of club guessing sequences were introduced by the author in [5].

Definition 2.5. Let $\tau : \omega_1 \to [\omega_1]^{\aleph_0}$ We say that a subset X of ω_1 is *τ-weakly tight* if and only if for every $\gamma \in \mathrm{nacc}(X)$, $X \cap \gamma \in \tau``\gamma$.

Definition 2.6. Let $\vec{C} = \langle C_\delta \,|\, \delta \in S\rangle$ be a tail club guessing sequence on a stationary subset S of $\omega_1 \cap \mathrm{Lim}$.

(i) We say that $\vec{C}$ is *τ-weakly tight* if and only if for every $\delta \in S$, C_δ is τ-weakly tight. We say that $\vec{C}$ is *weakly tight* if and only if there exists a function $\tau : \omega_1 \to [\omega_1]^{\aleph_0}$ such that $\vec{C}$ is τ-weakly tight.

(ii) We say that $\vec{C}$ is *simple* if and only if for every $\delta \in S$ and $\gamma \in C_\delta \cap S$, $C_\gamma \setminus C_\delta$ is unbounded in γ.

For example, if $\vec{C}$ has order type ε and ε is indecomposable, then $\vec{C}$ is simple. If $\vec{C}$ has order type ω, then $\vec{C}$ is also weakly tight. Simple weakly tight tail club guessing sequences are easier to deal with, as we will observe in Lemma 2.17 and Lemma 2.19.

Definition 2.7. A sequence $\langle N_\alpha \,|\, \alpha < \eta\rangle$ is called a *tower* if and only if

(i) for every limit $\alpha < \eta$, $N_\alpha = \bigcup_{\beta<\alpha} N_\beta$, and

(ii) for every $\alpha < \eta$, $\langle N_\beta \,|\, \beta \leq \alpha\rangle \in N_{\alpha+1}$.

Typically, $\eta \leq \omega_1$ and each N_α is a countable elementary submodel of $H(\theta)$ for some large regular cardinal θ. If it is the case, then for all $\beta < \alpha < \eta$, we have $N_\beta \subseteq N_\alpha$.

Fact 2.8. *Assume* CH *and let τ be a bijection from ω_1 onto $[\omega_1]^{\leq\aleph_0}$. Let $\varepsilon < \omega_1$ be a limit ordinal. Define $\bar{D}$ to be the set of all limit ordinals $\gamma < \omega_1$ such that there exists an unbounded subset C of γ that is τ-weakly tight and* $\mathrm{otp}(C) = \varepsilon$. *Note that $\bar{D}$ contains a club subset of ω_1.*

Proof. Let $\langle N_\alpha \mid \alpha < \omega_1\rangle$ be a tower of countable elementary submodels of $H(\aleph_2)$ with $\tau, \varepsilon \in N_0$. For each $\alpha < \omega_1$, let $\delta_\alpha = N_\alpha \cap \omega_1$. Let $\langle \varepsilon_n \mid n < \omega\rangle$ be a sequence of nonzero ordinals such that $\sum_{n<\omega} \varepsilon_n = \varepsilon$ and $\langle \varepsilon_n \mid n < \omega\rangle \in N_0$. Define D to be the set of all $\delta < \omega_1$ such that $N_\delta \cap \omega_1 = \delta$. It is easy to see that D is a club subset of ω_1.

We shall show that $D \subseteq \bar{D}$. Let $\gamma \in D$. Let $\langle \gamma_n \mid n < \omega\rangle$ be an increasing cofinal sequence in γ such that for each $n < \omega$, $\gamma_n + \varepsilon < \gamma_{n+1}$. Define

$$C = \bigcup_{n<\omega} \{\, \delta_\xi \mid \gamma_n \leq \xi \leq \gamma_n + \varepsilon_n \,\}$$

It is easy to see that C is an unbounded subset of $\delta_\gamma = \gamma$ and $\operatorname{otp}(C) = \sum_{n<\omega} \varepsilon_n = \varepsilon$. We shall show that C is τ-weakly tight. Let $\beta \in \operatorname{nacc}(C)$. Define $\bar\beta = \sup(C \cap \beta)$. By the definition of C, there exists $n < \omega$ and an ordinal ξ such that $\bar\beta = \delta_\xi$ and $\gamma_n \leq \xi \leq \gamma_n + \varepsilon_n$. Since $\langle N_\alpha \mid \alpha < \omega_1\rangle$ is a tower, $\langle N_\alpha \mid \alpha \leq \xi\rangle \in N_{\xi+1}$. So, $\langle \delta_\alpha \mid \alpha \leq \xi\rangle \in N_{\xi+1}$. Moreover, both $\langle \gamma_m \mid m \leq n\rangle$ and $\langle \varepsilon_m \mid m < \omega\rangle$ belong to $N_{\xi+1}$. Note

$$C \cap \beta = \bigcup_{m<n} \{\, \delta_\eta \mid \gamma_m \leq \eta \leq \gamma_m + \varepsilon_m \,\} \cup \{\, \delta_\eta \mid \gamma_n \leq \eta \leq \xi \,\}$$

So, $C \cap \beta$ can be computed correctly in $N_{\xi+1}$ and hence $C \cap \beta \in N_{\xi+1}$. Since τ is a bijection from ω_1 onto $[\omega_1]^{\leq\aleph_0}$, $C \cap \beta \in \tau``(N_{\xi+1} \cap \omega_1) = \tau``(\delta_{\xi+1}) \subseteq \tau``\beta$. □

Definition 2.9. Assume CH. Let τ, ε, and $\bar D$ as in Fact 2.8. Suppose that ε is indecomposable.

We shall define the forcing notion R to add a weakly tight tail club guessing sequence of order type ε be the set of all functions r such that $\operatorname{dom}(r) = \bar D \cap \delta$ for some ordinal $\delta < \omega_1$, and for every $\gamma \in \bar D \cap \delta$, $r(\gamma)$ is a τ-weakly tight closed unbounded subset of γ such that $\operatorname{otp}(r(\gamma)) = \varepsilon$. R is ordered by end-extension.

It is easy to observe that R is countably closed. We shall show that R indeed adds a weakly tight tail club guessing sequence of order type ε.

Definition 2.10. Let $\varepsilon < \omega_1$. A forcing notion P is called *ε-proper* if and only if whenever $\langle N_\alpha \,|\, \alpha < \varepsilon\rangle$ is a tower of countable elementary submodels of $H(\theta)$ for some sufficiently large regular cardinal θ with $\varepsilon, P \in N_0$, for every $p \in P \cap N_0$, there exists a $q \leq p$ that is (N_α, P)-generic for every $\alpha < \varepsilon$.

We say that P is $<\varepsilon$-proper if and only if P is ε'-proper for every $\varepsilon' < \varepsilon$.

Fact 2.11. *Every countably closed forcing notion is $<\omega_1$-proper.*

Lemma 2.12. *Let H be an R-generic filter over V. For each $\delta \in \bar D$, let $C_\delta = r(\delta)$ for some (all) $r \in H$. Then, $\vec C = \left\langle C_\delta \,\middle|\, \delta \in \bar D\right\rangle$ is a weakly tight tail club guessing sequence of order type ε on $\bar D$ in $V[H]$.*

Proof. Let $r \in R$. For each $\delta \in \bar D$, let $\dot C_\delta$ be an R-name for C_δ. Let $\dot D$ be an R-name for a club subset of ω_1.

Let $\langle N_\gamma \mid \gamma < \omega_1\rangle$ be a tower of countable elementary submodels of $H(\aleph_2)$ with $\tau, \varepsilon, \dot D \in N_0$. Let D be the set of all $\delta < \omega_1$ such that $N_\delta \cap \omega_1 = \delta$. Let $\gamma \in D$. By using the same argument as in the proof of Fact 2.8, we can build an unbounded subset C of γ such that $\operatorname{otp}(C) = \varepsilon$, C is τ-weakly tight, and $C \subseteq D$.

Since R is countably closed, R is $<\omega_1$-proper. So, there exists $r' \in R$ such that $r' \leq r$ and r' is (N_ξ, R)-generic for every $\xi \leq \gamma$. Since $\bar D \in N_0$, for every $\xi \leq \gamma$, $r' \Vdash N_\xi \cap \omega_1 \in \dot D$. In particular, $r' \Vdash C \subseteq D \subseteq \dot D$. Define $r'' = r' \cup \{\, \langle \gamma, C\rangle \,\}$. Then, $r'' \Vdash \dot C_\gamma = C \subseteq \dot D$. □

The following forcing notion was used in [9, Chapter XVIII §1] by S. Shelah.

Definition 2.13. Let $\vec C = \langle C_\delta \,|\, \delta \in S\rangle$ be a tail club guessing sequence on a stationary subset S of $\omega_1 \cap \mathrm{Lim}$. For every $X \in \mathrm{TCG}(\vec C)^+$, we define the *standard forcing $P(\vec C, X)$ to shoot a $\mathrm{TCG}(\vec C)$-measure one set through X* as follows: $p \in P(\vec C, X)$ if and only if p is a closed bounded subset of ω_1 such that for every $\delta \in S \cap \operatorname{acc}(p)$, $\{\, \delta \in S \cap \operatorname{acc}(p) \mid C_\delta \subseteq^* p \,\} \subseteq X$. $P(\vec C, X)$ is ordered by end-extension.

It is easy to see that $P(\vec{C}, X)$ forces that $\vec{C}$ is a tail club guessing sequence on ω_1 and $X \in \mathrm{TCG}(\vec{C})$.

We shall review several properties of this forcing notion.

Lemma 2.14 (S. Shelah [9])**.** *Let $\varepsilon < \omega_1$. Let $\left\langle P_\alpha, \dot{Q}_\beta \,\middle|\, \alpha \leq \eta, \beta < \eta \right\rangle$ be a countable-support iteration such that for each $\alpha < \eta$, P_α forces that $\dot{Q}_\alpha$ is ε-proper. Then P_η is ε-proper.*

The following notions were introduced by T. Eisworth and J. Roitman in [2].

Definition 2.15. Let P be a forcing notion. Let N be a countable elementary submodel of $H(\theta)$ for some sufficiently large regular cardinal θ with $P \in N$. We say that $p \in P$ is *totally (N, P)-generic* if and only if p is (N, P)-generic and decides all dense subsets of P lying in N.

We say that P is *totally proper* if and only if for every sufficiently large regular cardinal θ, every countable elementary submodel N of $H(\theta)$, and every $p \in N \cap P$, there exists a totally (N, P)-generic condition $q \leq p$.

The following proposition was proved in [2]

Proposition 2.16. *For every forcing notion P, the following are equivalent.*

(i) P is totally proper.

(ii) P is proper, and forcing with P adds no new function from ω to V.

(iii) P is proper, and forcing with P adds no new reals.

Unlike ε-properness, total properness is not preserved by countable support iteration. It was discussed by S. Shelah in [9].

The following lemma was proved by the author in [5].

Lemma 2.17. *Let $\vec{C} = \langle C_\delta \,|\, \delta \in S \rangle$ be a simple weakly tight tail club guessing sequence on a stationary subset S of $\omega_1 \cap \mathrm{Lim}$ and $X \in \mathrm{TCG}(\vec{C})^+$.*

(i) $P(\vec{C}, X)$ is totally proper.

(ii) If $\vec{C}$ has order type ε for some indecomposable ordinal ε, then $P(\vec{C}, X)$ is $<\varepsilon$-proper.

We can prove the following lemma about countable support iterations of forcing notions of this form.

Lemma 2.18. *Let $\vec{C} = \langle C_\delta \,|\, \delta \in S \rangle$ be a simple weakly tight tail club guessing sequence on a stationary subset S of $\omega_1 \cap \mathrm{Lim}$. Let $\left\langle P_\alpha, \dot{Q}_\beta \mid \alpha \leq \eta, \beta < \eta \right\rangle$ be a countable-support iteration such that for every $\beta < \eta$, there exists a P_β-name $\dot{X}_\beta$ for a subset of ω_1 such that $\mathbb{1}_{P_\beta} \Vdash \dot{Q}_\beta = P(\vec{C}, \dot{X}_\beta)$. Let $\mathcal{D}$ be the set of all $p \in P_\eta$ such that there exists $\delta < \omega_1$ such that for every $\beta \in \mathrm{supp}(p)$, $p \restriction \beta$ decides $p(\beta)$ and $\max(p(\beta)) = \delta$.*

Let N be a countable elementary submodel of $H(\theta)$ for some sufficiently large regular cardinal θ with $\vec{C}, P_\eta \in N$. Let $\delta = N \cap \omega_1$. Then, there exists $p' \leq p$ such that p' is totally (N, P_η)-generic and $p' \in \mathcal{D}$. In particular, P_η is totally proper and $\mathcal{D}$ is dense.

Proof. Let $\langle \mathcal{E}_n \mid n < \omega \rangle$ be an enumeration of all open dense subsets of P_η lying in N, and $\langle \beta_n \mid n < \omega \rangle$ an enumeration of $N \cap \eta$. By induction, we shall build a decreasing sequence $\langle p_n \mid n < \omega \rangle$ in $P_\eta \cap N$ as follows.

First suppose $\delta \notin S$. Then, let $p_0 = 0$ and for every $n < \omega$, let $p_{n+1} \leq p_n$ be so that $p_{n+1} \in \mathcal{E}_n \cap N$. Define p_ω so that $\mathrm{supp}(p_\omega) = N \cap \eta$ and for every $\beta \in \mathrm{supp}(p_\omega)$, $p_\omega \restriction \beta \Vdash p_\omega(\beta) = \bigcup_{n<\omega} p_n(\beta) \cup \{\delta\}$. It is easy to see that p_ω is totally (N, P_η)-generic. We can also prove that for every $\beta \in \mathrm{supp}(p_\omega)$, $p_\omega \restriction \beta$ decides $p_\omega(\beta)$ and $p_\omega \restriction \beta \Vdash \max(p_\omega(\beta)) = \delta$.

Suppose $\delta \in S$. Let $\langle \mathcal{E}_n \mid n < \omega \rangle$ be an enumeration of all open dense subsets of P_η lying in N, and $\langle \beta_n \mid n < \omega \rangle$ an enumeration of $N \cap \eta$. By induction, we shall build a decreasing sequence $\langle p_n \mid n < \omega \rangle$ in $P_\eta \cap N$ as follows. Let $p_0 = p$. Suppose that p_n is defined. Let $p'_n \leq p_n$ be so that $p'_n \in \mathcal{E}_n$ and for every $m \leq n$, $p'_n \restriction \beta_m$ decides $p'_n(\beta_m)$. We shall identify $p'_n(\beta_m)$ with its decided value. Let $\zeta_n = \max \bigcup_{m \leq n} p'_n(\beta_m)$ and $\gamma_n = \min(C_\delta \setminus \zeta_n)$. Since C_δ is an unbounded subset of δ, γ_n is defined.

Define $p_{n+1} \leq p'_n$ by letting

$$p_{n+1}(\beta) = \begin{cases} p'_n(\beta) \cup \{\gamma_n + 1\} & \text{if } \beta = \beta_m \text{ for some } m \leq n \\ p'_n(\beta) & \text{otherwise} \end{cases}$$

Define p_ω so that $\mathrm{supp}(p_\omega) = N \cap \eta$ and for every $\beta \in \mathrm{supp}(p_\omega)$, $p_\omega \restriction \beta \Vdash p_\omega(\beta) = \bigcup_{n<\omega} p_n(\beta) \cup \{\delta\}$. We shall show that $p_\omega \in P_\eta$. Since $|\mathrm{supp}(p_\omega)| = |N \cap \eta| = \aleph_0$, it suffices to show that for every $\beta < \eta$, $p_\omega \restriction \beta \Vdash p_\omega(\beta) \in \dot{Q}_\beta$. We go by induction on β. Suppose $p_\omega \restriction \beta \in P_\beta$. If $\beta \notin N \cap \eta$, then trivially $p_\omega \restriction \beta \Vdash p_\omega(\beta) \in \dot{Q}_\beta$. Suppose $\beta \in N \cap \eta$. Then, there exists $m < \omega$ such that $\beta = \beta_m$. Notice that for every $n, l < \omega$, if $m \leq n < l$, we have $\gamma_n \notin p_l(\beta_m)$. So, $p_\omega(\beta_m) \cap \{\gamma_n \mid m \leq n < \omega\} = \varnothing$ It is easy to see that $\langle \gamma_n \mid n < \omega \rangle$ is an unbounded subset of C_δ. So, $C_\delta \not\subseteq^* p_\omega(\beta_m)$ and hence $p_\omega \restriction \beta \Vdash p_\omega(\beta) \in \dot{Q}_\beta = P(\vec{C}, \dot{X}_\beta)$.

Clearly, p_ω satisfies the desired conditions. □

When $p \in \mathcal{D}$, we say that the *height of* p is δ if and only if for every $\beta \in \mathrm{supp}(p)$, $\max(p(\beta)) = \delta$, and we write $\mathrm{ht}(p) = \delta$. We may identify $p(\beta)$ with its decided value and identify p with a subset of $\eta \times (\delta + 1)$ by considering $\{\langle \beta, \gamma \rangle \mid \gamma \in p(\beta)\}$.

Lemma 2.19. *Let $\vec{C} = \langle C_\delta \mid \delta \in S \rangle$ be a simple weakly tight tail club guessing sequence on a stationary subset S of $\omega_1 \cap \mathrm{Lim}$ and $X \in \mathrm{TCG}(\vec{C})^+$. Suppose that θ is a sufficiently large regular cardinal. Suppose $\langle N_\gamma \mid \gamma < \nu + 1 \rangle$ is a tower of countable elementary submodels of $H(\theta)$. Define $\delta_\gamma = N_\gamma \cap \omega_1$ for all $\gamma \leq \nu$. Let D be a closed subset of $\nu + 1$ such that for every $\gamma \in \mathrm{nacc}(D)$, $D \cap \gamma \in N_\gamma$ and for every limit ordinal $\gamma \leq \nu$, if $\delta_\gamma \in S$, then $C_{\delta_\gamma} \not\subseteq^* \{\delta_\mu \mid \mu \in D \cap \gamma\}$*

Let $\left\langle P_\alpha, \dot{Q}_\beta \mid \alpha \leq \eta, \beta < \eta \right\rangle$ be a countable-support iteration such that for every $\alpha < \eta$, there exists a P_α-name $\dot{X}_\alpha$ for a subset of ω_1 such that $\mathbb{1}_{P\alpha} \Vdash \dot{Q}_\mu = P(\vec{C}, \dot{X}_\alpha)$.

Then, for every $p \in P_\eta \cap N_0$, there exists $q \leq p$ such that for every $\gamma \leq \nu$,

(i) if $\gamma \in D$, then q is totally (N_γ, P_η)-generic, and
(ii) if $\gamma \notin D$, then for every $\beta \in N_\gamma \cap \eta$, $\delta_\gamma \notin q(\beta)$.

Proof. Define $\mathcal{D}$ as in Lemma 2.18. Then, $\mathcal{D}$ is dense in P_η, so we may focus on $\mathcal{D}$. Let $\varphi(q, \gamma)$ denote the statement that 'if $\gamma \in D$, then q is totally $(N_\gamma, \mathcal{D})$-generic; if $\gamma \notin D$, then for every $\beta \in N_\gamma \cap \eta$, $\delta_\gamma \notin p(\beta)$'. Notice that if $\varphi(q, \gamma)$, $\delta_\gamma \leq \mathrm{ht}(q)$, and $q' \leq p$, then $\varphi(q', \gamma)$ also holds.

Claim 2.19.1. Let $q, q' \in \mathcal{D}$ and $\gamma < \nu$ be so that $q' \leq q$ and for every $\xi \leq \gamma$, $\varphi(q', \xi)$ holds. Then, there exists $\bar{q} \in P$ such that $\bar{q} \leq q$, $\mathrm{ht}(\bar{q}) \leq \delta_\gamma$, and for every $\xi \leq \gamma$, $\varphi(\bar{q}, \xi)$.

Proof. Let $\bar{\gamma} = \max(D \cap \gamma)$. If $\mathrm{ht}(q) \geq \delta_{\bar{\gamma}}$, then it is easy to observe that $\bar{q} = q$ works.

Suppose $\mathrm{ht}(q) < \delta_{\bar{\gamma}}$. Define $\bar{q}$ by letting $\mathrm{supp}(\bar{q}) = \mathrm{supp}(q')$ and for every $\beta \in \mathrm{supp}(\bar{q})$, $\bar{q}(\beta) = (q'(\beta) \cap (\delta_{\bar{\gamma}}+1)) \cup \{\, \delta_{\bar{\gamma}} + 1 \,\}$. Clearly, $\bar{q} \in \mathcal{D}$ and $\mathrm{ht}(\bar{q}) = \delta_{\bar{\gamma}+1} < \delta_\gamma$. Since $\mathrm{ht}(q) < \delta_{\bar{\gamma}}$, for every $\beta \in \mathrm{supp}(q)$, we have $q(\beta) = q'(\beta) \cap \delta_{\bar{\gamma}} = \bar{q}(\beta) \cap \delta_{\bar{\gamma}}$. Thus, $\bar{q} \leq q$. It is easy to see that for every $\xi \leq \gamma$, $\varphi(\bar{q}, \xi)$ holds. □

We go by induction on ν. For every $\gamma < \nu$, let $\delta_\gamma = N_\gamma \cap \omega_1$.

Case 2.19.1. $\nu = 0$.

If $0 \notin D$, then let $q = p$. If $0 \in D$, then since $\mathcal{D}$ is totally proper, there exists $q \leq p$ such that q is totally $(N_0, \mathcal{D})$-generic. It is easy to see that in both cases, q works.

From now on, assume $\nu > 0$.

Case 2.19.2. ν is a successor ordinal and $\nu \in D$.

Let $\bar{\nu}$ be the predecessor ordinal of ν. By the Inductive Hypothesis, there exists $\bar{q} \leq p$ such that $\varphi(\bar{q}, \gamma)$ holds for all $\gamma \leq \bar{\nu}$. Since $\nu \in \mathrm{nacc}(D)$, we have $\langle N_\gamma \mid \gamma \leq \bar{\nu} \rangle, D \cap \nu \in N_\nu$. So, without loss of generality, we may assume $\bar{q} \in N_\nu$ and $\mathrm{ht}(\bar{q}) \leq \delta_{\bar{\nu}}$. Let $\bar{q}' \in \mathcal{D}$ be defined by $\mathrm{supp}(\bar{q}') = N_{\bar{\nu}} \cap \eta$ and for every $\beta \in \mathrm{supp}(\bar{q}')$, let $\bar{q}'(\beta) = \bar{q}(\beta) \cup \{\, \delta_{\bar{\nu}} + 1 \,\}$. Since $\mathcal{D}$ is totally proper, there exists $q \leq \bar{q}'$ such that q is totally $(N_\nu, \mathcal{D})$-generic. It is easy to see that $\varphi(q, \gamma)$ holds for all $\gamma \leq \nu$.

Case 2.19.3. ν is a successor ordinal and $\nu \notin D$.

Let $\bar{\nu}$ be the predecessor ordinal of ν. By the Inductive Hypothesis, there exists $q \leq p$ such that $\varphi(q, \gamma)$ holds for all $\gamma \leq \bar{\nu}$. We may assume that $\mathrm{ht}(q) \leq \delta_{\bar{\nu}}$. So, $\varphi(q, \nu)$ holds also.

Case 2.19.4. ν is a limit ordinal and $\nu \notin D$.

Since D is closed, there exists $\bar{\nu} < \nu$ such that $D \subseteq \bar{\nu}$. By the Inductive Hypothesis, there exists $q \leq p$ such that $\varphi(q, \gamma)$ holds for all $\gamma \leq \bar{\nu}$ and $\mathrm{ht}(\bar{q}) \leq \delta_{\bar{\nu}}$. Since $\mathrm{ht}(\bar{q}) \leq \delta_{\bar{\nu}}$ and $D \cap [\bar{\nu}, \nu] = \varnothing$, we have for every $\gamma \in (\bar{\nu}, \nu]$, $\varphi(q, \gamma)$ holds.

Case 2.19.5. ν is a limit ordinal and $\nu \in \mathrm{nacc}(D)$.

Since $\nu \in \mathrm{nacc}(D)$, there exists the predecessor $\bar{\nu}$ of ν in D. Thus, $D \cap \nu = D \cap (\bar{\nu} + 1)$. Also, by assumption, we have $\langle N_\gamma \mid \gamma \leq \bar{\nu} \rangle, D \cap \nu \in N_\nu$. Since ν is a limit ordinal, we have $N_\nu = \bigcup_{\mu < \nu} N_\mu$. So, there exists $\mu < \nu$ such that $D \cap \nu \in N_\mu$. Without loss of generality, we may assume $\mu > \bar{\nu}$. Since $\langle N_\gamma \,|\, \gamma < \nu + 1 \rangle$ is a tower, we have $\langle N_\gamma \,|\, \gamma \leq \mu \rangle \in N_{\mu+1}$. Notice $D \cap (\delta_\mu + 1) = D \cap (\bar{\nu} + 1) \in N_{\mu+1}$. By the Inductive Hypothesis, there exists $\bar{q} \leq p$ such that $\varphi(\bar{q}, \gamma)$ holds for all $\gamma \leq \mu$, $\mathrm{ht}(\bar{q}) \leq \delta_\mu$ and $\bar{q} \in N_{\mu+1}$.

If $\delta_\nu \notin S$, let $\langle \xi_n \,|\, n < \omega \rangle$ be an increasing cofinal sequence in δ_ν such that $\xi_0 > \delta_{\mu+1}$ and for all $n < \omega$, there exists $\gamma < \nu$ such that $\xi_n < \delta_\gamma \leq \xi_{n+1}$. If $\delta_\nu \in S$, let $\langle \xi_n \,|\, n < \omega \rangle$ be an increasing cofinal sequence in C_{δ_ν} such that $\xi_0 > \delta_{\mu+1}$ and for all $n < \omega$, there exists $\gamma < \nu$ such that $\xi_n < \delta_\gamma \leq \xi_{n+1}$.

Since $\{ \delta_\gamma \mid \gamma < \nu \}$ is closed, for each $n < \omega$, there exists $\gamma_n < \nu$ such that $\delta_{\gamma_n} \leq \xi_n < \delta_{\gamma_n+1}$. By the definition of $\langle \xi_n \mid n < \omega \rangle$, $\langle \gamma_n \mid n < \omega \rangle$ is increasing. Let $\{ \mathcal{E}_n \mid n < \omega \}$ be an enumeration of all open dense subsets of $\mathcal{D}$ lying in N_ν such that for all $n < \omega$, $\mathcal{E}_n \in N_{\gamma_n}$.

We shall build a decreasing sequence $\langle q_n \mid n < \omega \rangle$ with $q_n \in N_{\gamma_n}$ as follows. Define $q_0 \in P$ so that $\operatorname{supp}(q_0) = N_\mu \cap \eta$ and for every $\beta \in \operatorname{supp}(q_0)$, $q_0(\beta) = \bar{q}(\beta) \cup \{ \delta_\mu + 1 \}$. Then, $q_0 \in N_{\mu+1}$. Since $\delta_{\mu+1} < \xi_0 < \delta_{\gamma_0+1}$, we have $\mu + 1 \leq \gamma_0$. So, $q_0 \in N_{\mu+1} \subseteq N_{\gamma_0}$. Since $q_0 \in N_{\mu+1}$, we have $\operatorname{ht}(q_0) < \delta_{\mu+1}$. So, for every $\gamma \in (\mu, \nu)$ and $\beta \in N_\gamma \cap \eta$, $\delta_\gamma \notin q_0(\beta)$.

Suppose that $q_n \in N_{\gamma_n}$ is defined. Define q_n' so that $\operatorname{supp}(q_n') = N_{\gamma_n} \cap \eta$ and for every $\beta \in \operatorname{supp}(q_n')$, $q_n'(\beta) = q_n(\beta) \cup \{ \xi_n + 1 \}$. Then $q_n' \in N_{\gamma_n+1}$. Since $q_n', \mathcal{E}_n \in N_{\gamma_n+1}$, there exists $q_{n+1} \leq q_n'$ such that $q_{n+1} \in \mathcal{E}_n \cap N_{\gamma_n+1}$. It is easy to see that for every $\beta \in N_{\gamma_n} \cap \eta$, $\xi_n \notin q_{n+1}(\beta)$, $\xi_n < \operatorname{ht}(q_{n+1})$, and for every $\gamma \in (\mu, \nu)$ and $\beta \in N_\gamma \cap \eta$, $\delta_\gamma \notin q_{n+1}(\beta)$. Since $\gamma_n + 1 \leq \gamma_{n+1}$, we have $q_{n+1} \in N_{\gamma_{n+1}}$.

Define $q \in \mathcal{D}$ so that $\operatorname{supp}(q) = N_\nu \cap \eta$ and for every $\beta \in \operatorname{supp}(q)$, $q(\beta) = \bigcup_{n<\omega} q_n(\beta) \cup \{ \delta_\nu \}$. If $\delta_\nu \notin S$, trivially we have $q \in \mathcal{D}$. Suppose $\delta_\nu \in S$. We shall show that for every $\beta \in \operatorname{supp}(q)$, $C_{\delta_\nu} \not\subseteq^* q(\beta)$. Since $\operatorname{supp}(q) = N_\nu \cap \eta$, there exists $n < \omega$ such that $\beta \in N_{\gamma_n} \cap \eta$. Then, for every $m < \omega$, if $n \leq m$, then $\beta \in N_{\gamma_m} \cap \eta$. So, $\xi_m \notin q_{m+1}(\beta)$ and $\xi_m < \operatorname{ht}(q_{m+1})$. Thus, $\xi_m \notin q(\beta)$. Therefore, $C_{\delta_\nu} \not\subseteq^* q(\beta)$. So, $q \in P$. For all $\gamma \leq \mu$, since $\varphi(q_0, \gamma)$ holds, $\delta_\gamma \leq \delta_\mu < \operatorname{ht}(q_0)$ and $q \leq q_0$, we have $\varphi(q, \gamma)$. For all $\gamma \in (\mu, \nu)$ and $\beta \in N_\gamma \cap \eta$, since $\gamma \notin D$ and $\delta_\gamma \notin q(\beta)$, $\varphi(q, \gamma)$. Since $q \in \mathcal{E}_n$ for all $n < \omega$, q is totally $(N_\nu, \mathcal{D})$-generic. Thus, $\varphi(q, \gamma)$ holds for all $\gamma \leq \nu$.

We shall prove the following claim before discussing the cases where $\nu \in \operatorname{acc}(D)$.

Claim 2.19.2. If $\nu \in \operatorname{acc}(D)$, then for every $\gamma < \nu$, $D \cap (\gamma + 1) \in N_{\min(D \setminus (\gamma+1))}$.

Proof. Let $\mu = \min(D \setminus (\gamma+1))$. Since $\nu \in \operatorname{acc}(D)$, we have $\mu < \nu$. So, $D \cap (\gamma+1) = D \cap \mu \in N_\mu$ by assumption. $\square$

Case 2.19.6. $\nu \in \operatorname{acc}(D)$ and $\delta_\nu \notin S$.

Let $\langle \nu_n \mid n < \omega \rangle$ be an increasing cofinal sequence in ν. We shall construct a decreasing sequence $\langle p_n \mid n < \omega \rangle$ and increasing sequences $\langle \gamma_n \mid n < \omega \rangle$, $\langle \mu_n \mid n < \omega \rangle$, and $\langle \xi_n \mid n < \omega \rangle$ of ordinals such that for all $n < \omega$, $p_n \in N_{\gamma_n+1}$, $\gamma_n \leq \mu_n < \gamma_{n+1}$, $\mu_n \geq \nu_n$, $\operatorname{ht}(p_n) \geq \delta_{\gamma_n}$, and $\varphi(p_n, \gamma)$ holds for all $\gamma \leq \gamma_n$. Let $\gamma_0 = 0$. By the Inductive Hypothesis, there exists $p' \leq p$ such that $p' \in N_1$ and $\varphi(p', 0)$ holds. Without loss of generality, we may assume $\operatorname{ht}(p') \leq \delta_0$. Define $p_0 \in P$ so that $\operatorname{supp}(p_0) = \operatorname{supp}(p') \cup (N_0 \cap \eta)$ and for every $\beta \in \operatorname{supp}(p_0)$, $p_0(\beta) = p'(\beta) \cup \{ \delta_0 + 1 \}$.

Suppose that p_n and γ_n have been defined. Let $\mu_n = \max \{ \nu_n, \gamma_n \}$. Let γ_{n+1} be the largest such that either $\gamma_{n+1} \leq \mu_n + 1$ or $D \cap (\mu_n + 1) \notin N_{\gamma_n+1}$. Since $\langle N_\gamma \mid \gamma \leq \nu \rangle$ is continuous, such γ_{n+1} exists. Trivially, we have $D \cap (\mu_n + 1) \in N_{\gamma_{n+1}+1}$. Since $D \cap (\mu_n + 1) \in N_{\min(D \setminus (\mu_n+1))}$, we have $\gamma_{n+1} < \min(D \setminus (\mu_n + 1))$. So, $(\mu_n, \gamma_{n+1}] \cap D = \varnothing$. By the Inductive Hypothesis, there exists $p_n' \leq p_n$ such that $\varphi(p_n', \gamma)$ holds for all $\gamma \in [\gamma_n + 1, \mu_n]$. Without loss of generality, we may assume $\operatorname{ht}(p_n') \leq \delta_{\mu_n}$ and $p_n' \in N_{\gamma_{n+1}+1}$. Define $p_{n+1} \in \mathcal{D}$ so that $\operatorname{supp}(p_{n+1}) = \operatorname{supp}(p_n') \cup (N_{\gamma_{n+1}} \cap \eta)$ and for every $\beta \in \operatorname{supp}(p_{n+1})$, $p_{n+1}(\beta) = p_n'(\beta) \cup \{ \delta_{\gamma_{n+1}} + 1 \})$. Then, $\varphi(p_{n+1}, \gamma)$ holds for all $\gamma \leq \mu_n$ and $\operatorname{ht}(p_{n+1}) = \delta_{\gamma_{n+1}+1}$. Since $\operatorname{ht} p_n' \leq \delta_{\mu_n}$, we have for all $\gamma \in (\mu_n, \gamma_{n+1}]$ and $\beta \in N_\gamma \cap \eta$, $\delta_\gamma \notin p_{n+1}(\beta)$. Since $(\mu_n, \gamma_{n+1}] \cap D = \varnothing$, $\varphi(p_{n+1}, \gamma)$ holds for all $\gamma \in (\mu_n, \gamma_{n+1}]$.

Define q so that $\mathrm{supp}(q) = \bigcup_{n<\omega} \mathrm{supp}(p_n)$ and for every $\beta \in \mathrm{supp}(q)$, $q(\beta) = \bigcup_{n<\omega} p_n(\beta) \cup \{ \delta_\nu \}$. Since $\delta_\nu \notin S$, we have $q \in \mathcal{D}$. Clearly, we have for all $\gamma < \nu$, $\varphi(q, \gamma)$ holds. Since $\nu \in \mathrm{acc}(D)$, there exist unboundedly many $\gamma < \nu$ such that $\gamma \in D$. Then, q is totally $(N_\gamma, \mathcal{D})$-generic. So, q is totally $(N_\nu, \mathcal{D})$-generic. Since $\nu \in D$, it implies that $\varphi(q, \nu)$ holds.

Case 2.19.7. $\nu \in \mathrm{acc}(D)$ and $C_{\delta_\nu} \subseteq^* \{ \delta_\gamma \mid \gamma < \nu \}$

Since $C_{\delta_\nu} \nsubseteq^* \{ \delta_\gamma \mid \gamma \in D \cap \nu \}$ by assumption, there exist unboundedly many $\gamma < \nu$ such that $\gamma \notin D$ and $\delta_\gamma \in C_{\delta_\nu}$.

Let $\langle \nu_n \,|\, n < \omega \rangle$ be an increasing cofinal sequence in ν. We shall construct a decreasing sequence $\langle p_n \,|\, n < \omega \rangle$ and increasing sequences $\langle \gamma_n \,|\, n < \omega \rangle$, $\langle \mu_n \,|\, n < \omega \rangle$, and $\langle \xi_n \,|\, n < \omega \rangle$ of ordinals such that for all $n < \omega$, $p_n \in N_{\gamma_n+1}$, $\gamma_n \leq \mu_n < \gamma_{n+1}$, $\mu_n \geq \nu_n$, $\mathrm{ht}(p_n) \geq \delta_{\gamma_n}$, and $\varphi(p_n, \gamma)$ holds for all $\gamma \leq \gamma_n$. Let $\gamma_0 = 0$. By the Inductive Hypothesis, there exists $p' \leq p$ such that $p' \in N_1$ and $\varphi(p', 0)$ holds. Without loss of generality, we may assume $\mathrm{ht}(p') \leq \delta_0$. Define p_0 so that $\mathrm{supp}(p_0) = \mathrm{supp}(p') \cup (N_0 \cup \eta)$ and for every $\beta \in \mathrm{supp}(p_0)$, $p_0(\beta) = p'(\beta) \cup \{ \delta_0 + 1 \}$.

Suppose that p_n and γ_n have been defined. Then, there exists $\mu_n < \nu$ such that $\mu_n \notin D$, $\delta_{\mu_n} \in C_{\delta_\nu}$, and $\mu_n \geq \max \{ \nu_n, \gamma_n \}$. Note $D \cap (\mu_n + 1) \in N_{\min(D \setminus (\mu_n+1))}$. Let γ_{n+1} be the largest such that either $\gamma_{n+1} \leq \mu_n + 1$ or $D \cap (\mu_n+1) \notin N_{\gamma_{n+1}}$. Since $\langle N_\gamma \,|\, \gamma \leq \nu \rangle$ is continuous, such an γ_{n+1} exists. Trivially, we have $D \cap (\mu_n + 1) \in N_{\gamma_{n+1}+1}$. Since $D \cap (\mu_n + 1) \in N_{\min(D \setminus (\mu_n+1))}$, we have $\gamma_{n+1} < \min(D \setminus (\mu_n + 1))$. So, $(\mu_n, \gamma_{n+1}] \cap D = \varnothing$. By the Inductive Hypothesis, there exists $p'_n \leq p_n$ such that $\varphi(p'_n, \gamma)$ holds for all $\gamma \in [\gamma_n + 1, \mu_n]$. In particular, since $\mu_n \notin D$ and $\varphi(p'_n, \mu_n)$ holds, for every $\beta \in N_{\mu_n} \cap \eta$, we have $\delta_{\mu_n} \notin p'_n(\beta)$. Without loss of generality, we may assume $\mathrm{ht}(p'_n) \leq \delta_{\mu_n}$ and $p'_n \in N_{\gamma_{n+1}+1}$. Define p_{n+1} so that $\mathrm{supp}(p_{n+1}) = \mathrm{supp}(p'_n) \cup (N_{\gamma_{n+1}} \cap \eta)$ and for every $\beta \in \mathrm{supp}(p_{n+1})$, $p_{n+1}(\beta) = p'_n(\beta) \cup \{ \delta_{\gamma_{n+1}} + 1 \}$. Then, $\varphi(p_{n+1}, \gamma)$ holds for all $\gamma \leq \mu_n$. Since $\mathrm{ht}(p'_n) \leq \delta_{\mu_n}$, we have for all $\gamma \in (\mu_n, \gamma_{n+1}]$ and $\beta \in N_\gamma \cap \eta$, $\delta_\gamma \notin p_{n+1}(\beta)$. Since $(\mu_n, \gamma_{n+1}] \cap D = \varnothing$, $\varphi(p_{n+1}, \gamma)$ holds for all $\gamma \in (\mu_n, \gamma_{n+1}]$.

Now, define q so that $\mathrm{supp}(q) = \bigcup_{n<\omega} \mathrm{supp}(p_n)$ and for every $\beta \in \mathrm{supp}(q)$, $q(\beta) = \bigcup_{n<\omega} p_n(\beta) \cup \{ \delta_\nu \}$. To show $q \in \mathcal{D}$, let $\beta \in \mathrm{supp}(q)$. It is easy to observe $\mathrm{supp}(q) = N_\nu \cap \eta$. Since $\langle \mu_n \mid n < \omega \rangle$ is cofinal in ν, for sufficiently large $n < \omega$, we have $\beta \in N_{\mu_n} \cap \eta$. By construction, for every $m < \omega$ with $m > n$, $\varphi(p_m, \mu_n)$ holds. If $\beta \in N_{\mu_n} \cap \eta$, since $\mu_n \notin D$, $\delta_{\mu_n} \notin p_m(\beta)$. Thus, $\delta_{\mu_n} \notin q(\beta)$. Since $\{ \mu_n \mid n < \omega \wedge \beta \in N_{\mu_n} \cap \eta \}$ is an unbounded subset of C_{δ_ν}, we have $C_{\delta_\nu} \nsubseteq^* q(\beta)$. Thus, $q \in P$. It is easy to see that for all $\gamma < \nu$, $\varphi(q, \gamma)$ holds. Thus, for all $\gamma \in D \cap \nu$, q is totally $(N_\gamma, \mathcal{D})$-generic. Since $D \cap \nu$ is unbounded in ν and $N_\nu = \bigcup_{\gamma<\nu} N_\gamma$, we can see that q is totally $(N_\nu, \mathcal{D})$-generic. Since $\nu \in D$, $\varphi(q, \nu)$ holds.

Case 2.19.8. $\nu \in \mathrm{acc}(D)$ and $C_{\delta_\nu} \nsubseteq^* \{ \delta_\gamma \mid \gamma < \nu \}$

Then, there exists unboundedly many $\xi < \delta_\nu$ such that $\xi \in C_{\delta_\nu}$ and $\xi \notin \{ \delta_\gamma \mid \gamma < \nu \}$. Let $\langle \nu_n \,|\, n < \omega \rangle$ be an increasing cofinal sequence in ν.

We shall construct a decreasing sequence $\langle p_n \,|\, n < \omega \rangle$ in $\mathcal{D}$, increasing sequences $\langle \gamma_n \,|\, n < \omega \rangle$, $\langle \mu_n \,|\, n < \omega \rangle$, and $\langle \xi_n \,|\, n < \omega \rangle$ such that for all $n < \omega$, $p_n \in N_{\gamma_n+1}$, $\xi_n \in C_{\delta_\nu} \setminus \{ \delta_\gamma \mid \gamma < \nu \}$, $\xi_n < \mathrm{ht}(p_{n+1})$, for every $\beta \in \mathrm{supp}(p_{n+1})$, $\xi_n \notin p_{n+1}(\beta)$, $\nu_n < \xi_n < \gamma_{n+1}$, and $\varphi(p_n, \gamma)$ holds for all $\gamma \leq \gamma_n$. Let $\gamma_0 = 0$. By the Inductive Hypothesis, there exists $p' \leq p$ such that $p' \in N_1$, $\varphi(p', 0)$ holds, and $\mathrm{ht}(p') \leq \delta_0$. Define $p_0 \in \mathcal{D}$ so that $\mathrm{supp}(p_0) = \mathrm{supp}(p') \cup (N_0 \cap \eta)$ and for every $\beta \in \mathrm{supp}(p_0)$, $p_0(\beta) = p'(\beta) \cup \{ \delta_0 + 1 \}$.

Suppose p_n and γ_n are defined. Let $\xi_n \in C_{\delta_\nu} \setminus \{ \delta_\gamma \mid \gamma < \nu \}$ such that $\xi_n > \max \{ \nu_n, \delta_{\gamma_n} \}$. Let μ_n be so that $\delta_{\mu_n} < \xi_n < \delta_{\mu_n+1}$. Let γ_{n+1} be the largest such that either $\gamma_{n+1} \leq \mu_n + 1$ or $D \cap (\mu_n + 1) \notin N_{\gamma_{n+1}}$. Then, $D \cap (\mu_n+1) \in N_{\gamma_{n+1}+1}$. Since $D \cap (\mu_n + 1) \in N_{\min(D \setminus (\mu_n+1))}$, we have $\gamma_{n+1} < \min(D \setminus (\mu_n + 1))$. So, $(\mu_n, \gamma_{n+1}] \cap D = \varnothing$. By the Inductive Hypothesis, there exists $p'_n \leq p_n$ such that $\varphi(p'_n, \gamma)$ holds for all $\gamma \in [\gamma_n + 1, \mu_n]$. Without loss of generality, we may assume $\mathrm{ht}(p'_n) \leq \delta_{\mu_n}$ and $p'_n \in N_{\gamma_{n+1}+1}$. Define p_{n+1} so that $\mathrm{supp}(p_{n+1}) = \mathrm{supp}(p'_n) \cup (N_{\gamma_{n+1}} \cap \eta)$ and for every $\beta \in \mathrm{supp}(p_{n+1})$, $p_{n+1}(\beta) = p'_n(\beta) \cup \{ \delta_{\gamma_{n+1}} + 1 \}$. Then, for every $\beta \in N_{\gamma_{n+1}} \cap \eta$, $(\delta_{\mu_n}, \delta_{\gamma_{n+1}}] \cap p_{n+1}(\beta) = \varnothing$. In particular, $\xi_n \notin p_{n+1}(\beta)$. Since $(\mu_n, \gamma_{n+1}] \cap D = \varnothing$ and for every $\beta \in N_{\gamma_{n+1}} \cap \eta$, $(\delta_{\mu_n}, \delta_{\gamma_{n+1}}] \cap p_{n+1}(\beta) = \varnothing$, $\varphi(p_{n+1}, \gamma)$ holds for all $\gamma \in (\mu_n, \gamma_{n+1}]$. Therefore, for all $\gamma \leq \gamma_{n+1}$, we have $\varphi(p_{n+1}, \gamma)$.

Define q so that $\mathrm{supp}(q) = \bigcup_{n<\omega} \mathrm{supp}(p_n)$ and for every $\beta \in \mathrm{supp}(q)$, $q(\beta) = \bigcup_{n<\omega} p_n(\beta) \cup \{ \delta_\nu \}$. To show $q \in \mathcal{D}$, we shall show that for every $\beta \in \mathrm{supp}(q)$, $C_{\delta_\nu} \not\subseteq^* q(\beta)$. It is easy to see $\mathrm{supp}(q) = N_\nu \cap \eta$. Since $\langle \gamma_n \mid n < \omega \rangle$ is unbounded in ν, there exists $n < \omega$ such that $\beta \in N_{\gamma_n+1} \cap \eta$. Then, for every $m < \omega$ with $m \geq n$, $\xi_m \notin p_{m+1}(\beta)$ and $\mathrm{ht}(p_{m+1}) > \delta_{\gamma_{m+1}} \geq \delta_{\mu_m+1} > \xi_m$. So, $\xi_m \notin q(\beta)$. Since $\{ \xi_m \mid m \geq n \}$ is an unbounded subset of C_{δ_ν}, we have $C_{\delta_\nu} \not\subseteq^* q(\beta)$. By the same argument as in the previous case, it is easy to see that $\varphi(q, \gamma)$ holds for all $\gamma \leq \nu$. □

The following notion is the same as what was called an outside club guessing sequence by M. Džamonja and S. Shelah in [1].

Definition 2.20. Let W be an inner model of V and κ an uncountable regular cardinal in W. Then, we say that a subset C of κ is a *fast club subset of κ over W* if and only if for every club subset D of κ lying in W, $C \subseteq^* D$.

The following property of a precipitous tail club guessing sequence is not directly used in the following sections, but it will help understand them, particularly Section 4.

Lemma 2.21. *Let $\vec{C} = \langle C_\delta \mid \delta \in S \rangle$ be a tail club guessing sequence on a stationary subset S of ω_1 such that $\mathrm{TCG}(\vec{C})$ is precipitous. Let U be $\mathcal{P}(\omega_1)/\mathrm{TCG}(\vec{C})$-generic over V, and $j : V \to M$ the induced generic elementary embedding. Then, $j(C)_\delta$ is a fast club subset of ω_1 over V.*

Proof. Let D be a club subset of ω_1 with $D \in V$. Then, $\{ \delta \in S \mid C_\delta \subseteq^* D \}$ is $\mathrm{TCG}(\vec{C})$-measure one. Thus, $\omega_1 \in j(\{ \delta \in S \mid C_\delta \subseteq^* D \})$, namely $j(C)_\delta \subseteq^* D$ □

3. Precipitous ideals in $L[U]$

In this section, we shall present several lemmas about models of the form $L[U]$ and its extensions. Note that the square brackets are used to denote generic extensions and relative constructibility. The following theorem of K. Kunen shows that any models of the form $L[U]$ are closely related.

Theorem 3.1 (K. Kunen [7])**.** *Let κ_1, κ_2 be cardinals with $\kappa_1 < \kappa_2$. Let U_1 and U_2 be so that U_i is a normal measure on κ_i in $L[U_i]$. Then, $L[U_2]$ is an iterated ultrapower of $L[U_1]$.*

The following two lemmas are about κ-complete normal precipitous ideals on κ for some uncountable regular cardinal κ in a generic extension of $L[U]$.

Lemma 3.2. *Let κ be a measurable cardinal, U a normal measure on κ, and $j : V \to M$ the induced elementary embedding. Suppose $V = L[U]$. Let P be a forcing notion and G a P-generic filter over V. Suppose that in $V[G]$, I is a κ-complete normal precipitous ideal on κ. Let W be $\mathcal{P}(\kappa)/I$-generic over $V[G]$ and let $k : V[G] \to M'$ be the generic elementary embedding induced by W. Then, $M' = L[k(U)][k(G)]$.*

Proof. Since $k(U)$ and $k(G)$ belong to M', we have $L[k(U)][k(G)] \subseteq M'$. To see $M' \subseteq L[k(U)][k(G)]$, let $x \in M'$. Then, there exists a function f in $V[G]$ such that $[f]_W = x$. So, there exists a function g such that for every $\gamma < \kappa$, $g(\gamma)$ is a P-name such that $g(\gamma) \in V$ and $(g(\gamma))^G = f(\gamma)$. So, in M', $[g]_W$ is a $k(P)$-name such that $([g]_W)^{k(G)} = (k(g)(\kappa))^{k(G)} = k(f)(\kappa) = [f]_W = x$. Since $g : \kappa \to V$, we have $[g]_W \in L[k(U)]$. Thus, $x \in L[k(U)][k(G)]$. □

Lemma 3.3. *Let κ be an uncountable cardinal. Then, for every $x \in H(\kappa^+)$, there exist functions ρ_x and ν_x with domain κ that satisfy the following condition. Suppose that $V[G]$ is a generic extension of V, κ and J is a κ-complete normal precipitous ideal J on κ in $V[G]$. Let W be $\mathcal{P}(\kappa)/J$-generic over $V[G]$ and $k : V[G] \to M'$ the induced generic elementary embedding. Then, we have $[\rho_x]_W = x$ and $[\nu_x]_W = k \restriction x$.*

Proof. We shall prove it by induction on the rank of x. Let $x \in H(\kappa^+)$ be so that for every $y \in x$, ρ_y and ν_y are defined. Let $f : |x| \to x$ be a bijection. Since $x \in H(\kappa^+)$, we have $|x| \leq \kappa$. If $|x| < \kappa$, then let $\rho_x(\gamma) = \{\rho_{f(\xi)}(\gamma) \mid \xi < |x|\}$. If $|x| = \kappa$, then let $\rho_x(\gamma) = \{\, \rho_{f(\xi)}(\gamma) \mid \xi < \gamma \,\}$. Define $\nu_x = \{\, \langle \rho_y(\gamma), y\rangle \mid y \in \rho_x(\gamma) \,\}$.

It is easy to verify that ρ_x and ν_x satisfy the desired conditions. □

The point of the last lemma is that ρ_x and ν_x are defined in V without specifying a precipitous ideal in the extension.

4. A precipitous tail club guessing ideal from a measurable cardinal

The following three sections are devoted to the proof of the following theorem.

Theorem 4.1. *Let κ be a measurable cardinal, U a normal measure on κ, and $\varepsilon < \kappa$ an indecomposable ordinal. Then, there is a forcing extension in which $\kappa = \omega_1$ and*

- *(i) there exists a tail club guessing sequence $\vec{C} = \langle C_\delta \mid \delta \in S\rangle$ of order type ε on a stationary subset S of $\omega_1 \cap \mathrm{Lim}$ such that $\mathrm{TCG}(\vec{C})$ is precipitous.*

Moreover, if the ground model satisfies $V = L[U]$, then the following hold in the extension:

- *(ii) NS_{ω_1} is nowhere precipitous, and*
- *(iii) for every weakly tight tail club guessing sequence $\vec{C}' = \langle C'_\delta \mid \delta \in S'\rangle$ of order type $<\varepsilon$ on a stationary subset S' of $\omega_1 \cap \mathrm{Lim}$, $\mathrm{TCG}(\vec{C}')$ is not precipitous.*

In this section, we shall construct the model that witnesses this theorem and also prove (i). Our strategy is similar to the one in [6]. Namely, we begin with a model with a measurable cardinal κ and collapse it to ω_1 by using Levy collapse to make a normal precipitous ideal on ω_1. Then, we shall modify it into a tail club guessing ideal while preserving its precipitousness. The major differences are as follows.

- After collapsing κ to ω_1, we generically add a weakly tight tail club guessing sequence $\vec{C} = \langle C_\delta \mid \delta \in S \rangle$ of order type ε. The fact that $\vec{C}$ is added generically at this point helps us go through the following arguments.
- We shall use the countable support iteration framework. This is a superficial difference because we shall define a dense subset $\mathcal{D}_{\alpha,\zeta}$ of the iteration Q_α that has a similar structure as in [6].
- Instead of shooting clubs in [6], we shall use the standard forcing to shoot a $\mathrm{TCG}(\vec{C})$-measure one set, defined in 2.13. Since this type of forcing notion is proper, we can use the general lemmas about the countable support iteration of proper forcing notions.
- Suppose that $\mathrm{TCG}(\vec{C})$ is precipitous in $V[G][H]$, U is $\mathcal{P}(\omega_1)/\,\mathrm{TCG}(\vec{C})$-generic over $V[G][H]$, and $j : V[G][H] \to N \subseteq V[G][H][U]$ is an induced generic elementary embedding. Then, by Lemma 2.21, $j(C)_{\omega_1}$ is fast club over $V[G][H]$. The construction is designed to make this happen.

Let κ be a measurable cardinal and $\varepsilon < \kappa$ an indecomposable ordinal. Without loss of generality, we may assume $2^\kappa = \kappa^+$. We shall construct a forcing extension in which there exists a precipitous tail club guessing sequence on ω_1 of order type ε whose associated ideal is precipitous.

Let U be a normal measure on κ, and $j : V \to M$ the elementary embedding induced by U. Let $P = \mathrm{Coll}(\omega, {<}\kappa)$. Let G be P-generic over V and $\hat{G}$ $j(P)$-generic over V extending G. It is well known that j can be extended to $j_0 : V[G] \to M[\hat{G}]$ by letting $j_0(\dot{x}^G) = j(\dot{x})^{\hat{G}}$ for every P-name $\dot{x}$. Let $\dot{j}_0$ be a $j(P)/G$-name for j_0. Work in $V[G]$. Note $\omega_1 = \kappa$ in this model. Define an ideal I_0 on κ by: $X \in I_0$ if and only if $\mathbb{1}_{j(P)/G} \Vdash$ '$\kappa \notin \dot{j}_0(X)$'. It is also well-known that if we define $\pi_0 : \mathcal{P}(\omega_1)/I_0 \to \mathcal{B}(j(P)/G)$ by $\pi_0(X) = [\![\kappa \in \dot{j}_0(X)]\!]$, then π_0 is a dense embedding and hence $j(P)/G \simeq \mathcal{P}(\kappa)/I_0$. Note $(M[G])^\kappa \cap V[G] \subseteq M[G]$.

In $V[G]$, we shall define a countable support iteration $\left\langle Q_\alpha, \dot{R}_\beta \,\middle|\, \alpha \leq \omega_2, \beta < \omega_2 \right\rangle$ by induction so that for every $\alpha < \omega_2$, Q_α forces that $|\dot{R}_\alpha| = \aleph_1$ and $\dot{R}_\alpha$ is ${<}\varepsilon$-proper and totally proper.

During the inductive construction, we shall also define for each $\alpha < \omega_2$,

- a Q_α-name $\dot{X}_\alpha$ for a subset of ω_1,
- a Q_α-name $\dot{I}_\alpha$ for a normal ideal on ω_1, and
- a dense subset $\mathcal{D}_{\alpha,\zeta}$ of Q_α for each $\zeta < \omega_1$.

R_0 is the forcing notion to add a weakly tight tail club guessing sequence $\vec{C}$ of order type ε. For every $\alpha \in [1, \omega_2)$, Q_α forces that $\dot{R}_\alpha$ is either the trivial forcing notion or equal to $P(\vec{C}, \kappa \setminus \dot{X}_\alpha)$. Let $\dot{D}_\alpha$ be the $Q_{\alpha+1}$-name for the club subset added by $\dot{R}_\alpha$ when $\dot{R}_\alpha$ is forced to be $P(\vec{C}, \kappa \setminus \dot{X}_\alpha)$. We can show that for every $\alpha < \omega_2$ and $\zeta < \omega_1$, $|\mathcal{D}_{\alpha,\zeta}| = \aleph_1$. By Lemma 2.14, it follows that for every $\beta < \alpha \leq \omega_2$ and Q_β-generic filter H_β over $V[G]$, Q_α/H_β is ${<}\varepsilon$-proper. In addition, we will show that Q_α is totally proper. Since $2^\kappa = \kappa^+$ in V, we can show $2^{\aleph_1} = \aleph_2$ in $V[G]$. So, there exists a bookkeeping $\left\langle \dot{X}_\alpha \,\middle|\, \alpha < \omega_2 \right\rangle$ of all subsets of ω_1 in the extension of $V[G]$ by Q_{κ^+} so that every subset appears unboundedly many times. Since CH holds in $V[G]$, we can pick a bijection $\tau : \omega_1 \to [\omega_1]^{\aleph_0}$. Since Q_α is totally proper for every $\alpha < \omega_2$, τ remains a bijection from ω_1 onto $[\omega_1]^{\aleph_0}$ in the extension of $V[G]$ by Q_α. Let $\bar{D}$ be the set of all limit ordinals $\gamma < \omega_1$ such that there exists an unbounded subset C of γ that is τ-weakly tight and $\operatorname{otp}(C) = \varepsilon$. By Fact 2.8, $\bar{D}$ contains a club subset of ω_1.

At the zeroth stage, let R_0 be the forcing to add a weakly tight tail club guessing sequence of order type ε, defined in Definition 2.9, by using τ. If H_0 is R_0-generic over $V[G]$, for every $\delta \in \bar{D}$, let $C_\delta = r(\delta)$ for some (all) $r \in H_0$ with $\delta \in \operatorname{dom}(r)$. Define $\vec{C} = \langle C_\delta \mid \delta \in \bar{D} \rangle$. It is easy to see that $\vec{C}$ is a weakly tight tail club guessing sequence of order type ε on ω_1 in $V[G][H_0]$. It automatically defines Q_1. Let $\dot{\vec{C}}$ be a Q_1-name of $\vec{C}$. It is easy to observe that $|Q_1| = |R_0| = \aleph_1$.

Lemma 4.2. *Work in $V[G]$. Let $\hat{G}$ be a $j(P)$-generic filter over V extending G. Then, for every $q \in Q_1$, there exists a Q_1-generic filter $H_1' \in M[\hat{G}]$ over $V[G]$ with $q \in H_1'$.*

Proof. Note $\mathcal{P}(\aleph_1)^{V[G]} = \mathcal{P}(\aleph_1)^{M[G]}$ and $\left|\mathcal{P}(\aleph_1)^{M[G]}\right| = \aleph_0$ in $M[\hat{G}]$. So, in $V[\hat{G}]$, we have $|\mathcal{P}(Q_1)^{V[G]}| = \aleph_0$. Thus, we can easily generate a generic sequence beginning with q to build a Q_1-generic filter H_1' over $V[G]$. Since $(M[\hat{G}])^{\omega_1} \cap V[G] \subseteq M[\hat{G}]$, we have $H_1' \in M[\hat{G}]$. $\square$

We shall keep using Q_α-generic filters that are generated in $V[\hat{G}]$ as in the previous lemma. To distinguish such filters from the generic filter obtained by forcing, we shall add primes such as H_α' or H_α''.

Lemma 4.3. *Work in $V[G]$. Let $\hat{G}$ be a $j(P)$-generic filter over V extending G. Let $H_1' \in M[\hat{G}]$ a Q_1-generic filter over $V[G]$. Define $m_1 \in j_0(Q_1)$ by $m_1(0) = \bigcup \{ q(0) \mid q \in H_1' \}$. Let $\hat{H}_1$ be a $j_0(Q_1)$-generic filter over $V[\hat{G}]$ with $m_1 \in \hat{H}_1$. Define $j_1 : V[G][H_1'] \to M[\hat{G}][\hat{H}_1]$ by $j_1(\dot{x}^{G*H_1'}) = (j_0(\dot{x}))^{\hat{G}*\hat{H}_1}$. Then,*

(i) ${j_0}^{-1}\hat{H}_1 = H_1'$, and
(ii) j_1 is an elementary embedding.

Proof. Notice that $j_0 \restriction Q_1$ is an identity. So, m_1 is clearly a lower bound of j_0“H_1'. Thus, ${j_0}^{-1}\hat{H}_1 = H_1'$.

Let φ be a formula and $x_0, x_1, \ldots, x_{n-1} \in V[G][H_1']$ such that $\varphi(x_0, x_1, \ldots, x_{n-1})$ holds in $V[G][H_1]$. Let $\dot{x}_i$ be a Q_1-name for x_i for each $i < n$. Then, there exists $q \in Q_1$ such that $q \Vdash_{Q_1} \varphi(\dot{x}_0, \dot{x}_1, \ldots, \dot{x}_{n-1})$. Since j_0 is an elementary embedding,

$$q \Vdash_{Q_1} \varphi(\dot{x}_0, \dot{x}_1, \ldots, \dot{x}_{n-1}) \iff j_1(q) \Vdash_{j_0(Q_1)} \varphi(j_0(\dot{x}_0), j_0(\dot{x}_1), \ldots, j_0(\dot{x}_{n-1}))$$

Since $m_1 \in \hat{H}_1$, we have j_0“$H_1' \subseteq \hat{H}_1$. In particular, $j_1(q) \in \hat{H}_1$. So, in $M[\hat{G}][\hat{H}_1]$, we have

$$\varphi((j_0(\dot{x}_0))^{\hat{G}*\hat{H}_1}, (j_0(\dot{x}_1))^{\hat{G}*\hat{H}_1}, \ldots, (j_0(\dot{x}_{n-1}))^{\hat{G}*\hat{H}_1})$$

By the definition of j_1, this is equivalent to $\varphi(j_1(x_0), j_1(x_1), \ldots, j_1(x_{n-1}))$. $\square$

Lemma 4.4. *Work in $V[G]$. Let $\hat{G}$ be a $j(P)$-generic filter over V extending G and H_1 a Q_1-generic filter over $V[G]$ lying in $M[\hat{G}]$. Then, there exists an unbounded subset $C \in M[\hat{G}]$ of κ such that $\operatorname{otp}(C) = \varepsilon$, C is a τ-weakly tight fast club subset of κ over $V[G][H_1]$.*

Proof. We have $\mathcal{P}(\kappa)^{V[G][H_1]} = \mathcal{P}(\kappa)^{M[G][H_1]}$ and $\left|\mathcal{P}(\kappa)^{M[G][H_1]}\right| = \aleph_0$ in $M[\hat{G}]$. So, we can enumerate all club subsets of κ lying in $V[G][H_1]$ as $\langle E_n \mid n < \omega \rangle$ in $M[\hat{G}]$. Now, it is easy to build a C witnessing the claim. $\square$

Suppose that we have defined Q_α for some $\alpha \in [1, \kappa^+)$. Let H_α be Q_α-generic over $V[G]$. Since Q_α has a dense subset of size $\aleph_1$ and $(M[G])^{\omega_1} \cap V[G] \subseteq M[G]$, every Q_α-name for a subset of ω_1 has an equivalent name lying in $M[G]$.

We say that $q \in Q_\alpha$ is a *flat condition of height* ζ if and only if for every $\beta \in [1, \alpha) \cap \operatorname{supp}(q)$, $q \restriction \beta$ decides $q(\beta)$ and either $q(\beta) = \varnothing$ or $\max(q(\beta)) = \zeta$. When q is a flat condition, let $\operatorname{ht}(q)$ denote the height of q. For each $\zeta < \kappa$, let $\mathcal{D}_{\alpha,\zeta}$ be the set of all flat conditions in Q_α of height $\geq \zeta$. It is easy to see that $|\mathcal{D}_{\alpha,\zeta}| = \kappa$.

If $q \in Q_\alpha$ is a flat condition, then by definition for every $\beta < \operatorname{supp}(q)$, $q \restriction \beta$ decides $q(\beta)$. So, we identify $q(\beta)$ and its decided value by $q \restriction \beta$.

By using Lemma 2.18, we can easily show the following lemma.

Lemma 4.5.

(i) Q_α is totally proper, and
(ii) for every $\zeta < \kappa$, $\mathcal{D}_{\alpha,\zeta}$ is dense in Q_α.

Proof. For (i), Q_1 is countably closed and hence totally proper. Notice that for every $\beta \in [1, \alpha)$, $\dot{R}_\beta$ is forced by Q_β to be either the trivial forcing notion or the forcing notion of the form $P(\vec{C}, \dot{X}_\beta)$ where $\dot{X}_\beta$ is a Q_β-name for a subset of ω_1. By Lemma 2.18, Q_1 forces that $\dot{Q}_{1,\alpha}$ is totally proper. Therefore, Q_α is totally proper.

(ii) can be easily proved by using Lemma 2.18. □

Since $\mathcal{D}_{\alpha,0}$ is dense in Q_α, we shall identify Q_α with $\mathcal{D}_{\alpha,0}$ from now on.

Lemma 4.6. *Let $\hat{G}$ be $j(P)$-generic over V extending G. Let $\beta < \alpha < \omega_2$, $H_\beta \in V[\hat{G}]$ a Q_β-generic filter over $V[G]$, and $C \in V[\hat{G}]$ a τ-weakly tight fast club subset of κ over $V[G][H_\beta]$ of order type ε. Then, for every $q \in Q_\alpha$ with $q \restriction \beta \in H_\beta$, there exists a Q_α-generic filter H_α over $V[G]$ such that $H_\alpha \in M[\hat{G}]$, $H_\beta = H_\alpha \cap Q_\beta$, $q \in H_\alpha$, and C is a fast club subset of κ over $V[G][H_\alpha]$.*

Proof. In $V[\hat{G}]$, $\left|\mathcal{P}(Q_\alpha)^{V[G]}\right| = \aleph_0$. So, we can pick an enumeration $\langle \mathcal{E}_n \,|\, n < \omega \rangle \in V[\hat{G}]$ of all dense open subset of Q_α / H_β lying in $V[G][H_\beta]$. Since in $V[G]$, Q_α forces that $2^\kappa = \kappa^+ = (\kappa^+)^{V[G]}$ and $V[\hat{G}]$ satisfies $|(\kappa^+)^{V[G]}|^{M[\hat{G}]} = \aleph_0$, there exists an enumeration $\left\langle \dot{D}_n \,\middle|\, n < \omega \right\rangle$ of all Q_α / H_β-names for a club subset of κ.

Work in $V[\hat{G}]$. Let θ be a sufficiently large regular cardinal. We shall construct a sequence $\langle N_{n,\xi} \,|\, n < \omega \text{ and } \xi < \omega_1 \rangle$ so that for every $n < \omega$, $\langle N_{n,\xi} \,|\, \xi < \omega_1 \rangle \in V[G][H_\beta]$. In $V[G][H_\beta]$, pick a tower $\langle N_{0,\xi} \,|\, \xi < \omega_1 \rangle$ of countable elementary submodels of $H(\theta)^{V[G][H_\beta]}$ with $Q_\alpha / H_\beta, q, \mathcal{E}_0, \dot{D}_0 \in N_{0,0}$. Suppose that we have defined $\langle N_{n,\xi} \,|\, \xi < \omega_1 \rangle$. In $V[G][H_\beta]$, pick a tower $\langle N_{n+1,\xi} \,|\, \xi < \omega_1 \rangle$ of countable elementary submodels of $H(\theta)^{M[G][H_\beta]}$ such that $\mathcal{E}_{n+1}, \dot{D}_{n+1} \in N_{n+1,0}$ and for every $\xi < \omega_1$, $N_{n,\xi} \subseteq N_{n+1,\xi}$. For each $n < \omega$, define $E_n = \{ \xi < \omega_1 \mid N_{n,\xi} \cap \omega_1 = \xi \}$. Then, E_n is a club subset of ω_1 lying in $M[G][H_\beta]$. By assumption, $C \subseteq^* E_n$. Let $\zeta_n \in E_n$ be so that $C \setminus \zeta_n \subseteq E_n$. Without loss of generality, we may assume that $\langle \zeta_n \,|\, n < \omega \rangle$ is increasing.

Keep working in $V[\hat{G}]$. We shall build a decreasing sequence $\langle q_n \,|\, n < \omega \rangle$ in Q_α / H_β as follows. Let $q_0 = q$. Suppose that we have defined q_n so that $q_n \in N_{n,\zeta_n}$. Since $\operatorname{otp}(C) = \varepsilon$, $C \cap [\zeta_n, \zeta_{n+1})$ has order type $<\varepsilon$. Since Q_α / H_β is $<\varepsilon$-proper, there exists a $q_{n+1} \leq q_n$ such that $q_{n+1} \in \mathcal{E}_n \cap N_{n,\zeta_{n+1}}$ and for every $\gamma \in C \cap [\zeta_n, \zeta_{n+1})$, q_{n+1} is $(N_{n,\gamma}, Q_\alpha / H_\beta)$-generic. Notice that $q_{n+1} \in N_{n,\zeta_{n+1}} \subseteq N_{n+1,\zeta_{n+1}}$. Define $H_{\beta,\alpha}$ be the filter generated by $\{ q_n \mid n < \omega \}$. Let H_α be defined by $q' \in H_\alpha$ if and only if $q' \upharpoonright \beta \in H_\beta$ and $q' \upharpoonright [\beta, \alpha)^{V[G][H_\beta]} \in H_{\beta,\alpha}$. Since $M[\hat{G}]^{\aleph_1} \cap V[\hat{G}] \subseteq M[\hat{G}]$, we have $H_\alpha \in M[\hat{G}]$. It is easy to see that H_α satisfies the desired conditions. $\square$

By using the previous lemma with Lemma 4.4, we can observe that there exist a Q_α-generic filter $H_\alpha \in M[\hat{G}]$ over $V[G]$ and a τ-weakly tight fast club subset of ω_1 over $V[G][H_\alpha]$ of order type ε.

Note that since $|Q_\alpha| = \aleph_1$, $\mathcal{P}(Q_\alpha)^{V[G]} = \mathcal{P}(Q_\alpha)^{M[G]}$, so for every $H'_\alpha \subseteq Q_\alpha$, H'_α is Q_α-generic over $V[G]$ if and only if H'_α is Q_α-generic over $M[G]$. Moreover, if H'_α is a Q_α-generic over $V[G]$, we have $\mathcal{P}(\omega_1)^{V[G][H'_\alpha]} = \mathcal{P}(\omega_1)^{M[G][H'_\alpha]}$, so for every $C \subseteq \omega_1$, C is a fast club subset of ω_1 over $V[G][H'_\alpha]$ if and only if C is a fast club subset of ω_1 over $M[G][H'_\alpha]$. So, we can correctly describe these two properties in $M[\hat{G}]$.

Let $\hat{G}$ be a $j(P)$-generic filter over V extending G. Work in $M[\hat{G}]$. Let $H'_\alpha \in M[\hat{G}]$ be a Q_α-generic filter over $V[G]$, $H'_\beta = H'_\alpha \cap Q_\beta$ for all $\beta < \alpha$, and C' a τ-weakly tight fast club subset of κ over $V[G][H'_\alpha]$ of order type ε. Define $m'_\alpha = m'_\alpha(H'_\alpha, C')$ by for every $\beta < j(\alpha)$,

$$m'_\alpha(\beta) = \begin{cases} \bigcup \{\, q(0) \mid q \in H'_1 \,\} \cup \{\, \langle \kappa, C \rangle \,\} & \text{if } \beta = 0 \\ \varnothing & \text{if } \beta \notin j``\alpha \\ \varnothing & \text{if } \beta = j(\bar{\beta}) \text{ and } (\dot{X}_{\bar{\beta}})^{H'_{\bar{\beta}}} \notin (\dot{I}_{\bar{\beta}})^{H'_{\bar{\beta}}} \\ (\dot{D}_{\bar{\beta}})^{H'_{\bar{\beta}}} \cup \{\, \kappa \,\} & \text{if } \beta = j(\bar{\beta}) \text{ and } (\dot{X}_{\bar{\beta}})^{H'_{\bar{\beta}}} \in (\dot{I}_{\bar{\beta}})^{H'_{\bar{\beta}}} \end{cases}$$

Let $\dot{m}'_\alpha$ be a $j(P)$-name for m'_α.

Later in Lemma 4.7, we will show that $m'_\alpha(H'_\alpha, C') \in j_0(Q_\alpha)$, and for every $j_0(Q_\alpha)$-generic filter $\hat{H}_\alpha$ with $m'_\alpha(H'_\alpha, C') \in \hat{H}_\alpha$ we have ${j_0}^{-1}\hat{H}_\alpha = H'_\alpha$. Define $j_\alpha : V[G][H'_\alpha] \to M[\hat{G}][\hat{H}_\alpha]$ by for all Q_α-name $\dot{x}$, $j_\alpha(\dot{x}) = (j_0(\dot{x}))^{\hat{G} * \hat{H}_\alpha}$. Then, j_α is an elementary embedding. Let $\dot{j}_\alpha$ be the $j_0(Q_\alpha)$-name for j_α.

Work in $V[G][H_\alpha]$. Define I_α by: $X \in I_\alpha$ if and only if whenever $\dot{X}$ is a Q_α-name for X lying in $V[G]$, there exist $p \in G$ and $q \in H_\alpha$ such that $j(p)$ forces in $j(P)$ that for all Q_α-generic filters H'_α over $V[\dot{G}]$ with $q \in H'_\alpha$ and all τ-weakly tight fast club subset C of κ over $V[\dot{G}][H'_\alpha]$ of order type ε, $\dot{m}'_\alpha(H'_\alpha, C) \Vdash_{j_0(Q_\alpha)} \kappa \notin \dot{j}_\alpha(X')$ where $X' = \dot{X}^{H'_\alpha}$.

We shall show that the definition of I_α does not depend on the choice of $\dot{X}$. Work in $V[G]$. Suppose that $\dot{X}_1$ and $\dot{X}_2$ are both Q-names for X. Then, there exists $q_0 \in H_\alpha$ such that $q_0 \Vdash \dot{X}_1 = \dot{X}_2$. Suppose that there exist $p \in G$ and $q_1 \in H_\alpha$ such that $j(p)$ forces in $j(P)$ that for all Q_α-generic filters H'_α over $V[\dot{G}]$ with $q_1 \in H'_\alpha$ and all τ-weakly tight fast club subset C' of κ over $V[\dot{G}][H'_\alpha]$ of order type ε, $\dot{m}'_\alpha(H'_\alpha, C') \Vdash_{j_0(Q_\alpha)} \kappa \notin \dot{j}_\alpha(X')$ where $X' = \dot{X}_1^{H'_\alpha}$. Let $q_2 \in H_\alpha$ be the common extension of q_0 and q_1. We shall show that p and q_2 witnesses $\dot{X}_2^{H_\alpha} \in I_\alpha$. Let $\hat{G}$ be a $j(P)$-generic filter over V with $j(p) \in \hat{G}$ and work in $V[\hat{G}]$. Let $G' = \hat{G} \cap P$. Let H'_α be a Q_α-generic filter over $V[G']$ with $q_2 \in H'_\alpha$ and C a τ-weakly tight fast club subset of κ over $V[G'][H'_\alpha]$ of order type ε. Notice that since $q_2 \leq q_1$, we have $q_1 \in H'_\alpha$. Since $j(p) \in \hat{G}$, we have $m'_\alpha(H'_\alpha, C') \Vdash_{j_0(Q_\alpha)} \kappa \notin \dot{j}_\alpha(X')$ where $X' = \dot{X}_1^{H'_\alpha}$. Since $q_2 \leq q_0$, we have $q_0 \in H'_\alpha$. So, $X' = \dot{X}_2^{H'_\alpha}$. Thus, $X \in I_\alpha$ is witnessed by $\dot{X}_2$ also.

If $X_\alpha \notin I_\alpha$, then let $R_\alpha = \{\, \varnothing \,\}$, i.e. the trivial forcing notion. If $X_\alpha \in I_\alpha$, then let $R_\alpha = P(\vec{C}, \kappa \setminus X_\alpha)$. This completes the definition of $\left\langle Q_\alpha, \dot{R}_\beta \,\middle|\, \alpha \leq \omega_2, \beta < \omega_2 \right\rangle$

and $\left\langle \dot{I}_\alpha \,\middle|\, \alpha < \omega_2 \right\rangle$. For each $\alpha \in [1, \omega_2)$, let $\dot{D}_\alpha$ be a $Q_{\alpha+1}$-name for the club subset of κ added by $\dot{R}_\alpha$ if $\dot{R}_\alpha$ is non-trivial.

Now, we shall prove that $m'_\alpha \in j_0(Q_\alpha)$ in $V[\hat{G}]$.

Lemma 4.7. *The following hold in $V[G]$. Suppose that $\hat{G}$ is a $j(P)$-generic filter over V extending G and work in $V[\hat{G}]$. Let $H'_\alpha \in M[\hat{G}]$ be a Q_α-generic filter over $V[G]$, and $C' \in V[\hat{G}]$ be a τ-weakly tight fast club subset of κ over $V[G][H'_\alpha]$ of order type ε. Then, $m'_\alpha = m'_\alpha(H'_\alpha, C') \in j_0(Q_\alpha)$ and for every $j_0(Q_\alpha)$-generic filter $\hat{H}_\alpha$ over $M[\hat{G}]$, $j_1(C)_\kappa = C'$ and $j_0{}^{-1}\hat{H}_\alpha = H'_\alpha$ if and only if $m'_\alpha \in \hat{H}_\alpha$.*

Proof. Go by induction on α. When $\alpha = 1$, it is clear from Lemma 4.3. Suppose that $\alpha > 1$ and the conclusion holds for every $\beta < \alpha$.

Let $\hat{G}$, H'_α, and C' be as in the assumption. Consider $m'_\alpha = m'_\alpha(H'_\alpha, C')$. For each $\beta < \alpha$, let $H'_\beta = H'_\alpha \cap Q_\beta$.

Claim 4.7.1. $m'_\alpha \in j_0(Q_\alpha)$.

Proof. We have $\operatorname{supp}(m'_\alpha) \subseteq j``\alpha$, which is countable in $M[\hat{G}]$. By induction on $\beta < j(\alpha)$, we shall show that $m'_\alpha \restriction \beta \in j_0(Q)_\beta$. When $\beta = 0$ or $\beta = 1$, it is trivial. If $\beta < j(\alpha)$ is a limit ordinal and for every $\gamma < \beta$, $m'_\alpha \restriction \gamma \in j_0(Q)_\gamma$, since $\operatorname{supp}(m'_\alpha)$ is countable, we have $m'_\alpha \restriction \beta \in j_0(Q)_\beta$.

Suppose that $m'_\alpha \restriction \beta \in j_0(Q)_\beta$. We shall show $m'_\alpha \restriction (\beta+1) \in j_0(Q)_{\beta+1}$. If $\beta \notin j``\alpha$, then since $\beta \notin \operatorname{supp}(m'_\alpha)$, trivially we have $m'_\alpha \restriction (\beta+1) \in j_0(Q)_{\beta+1}$. Suppose $\beta \in j``\alpha$ and let $\bar{\beta} < \alpha$ be so that $\beta = j(\bar{\beta})$. If $(\dot{X}_{\bar{\beta}})^{H'_{\bar{\beta}}} \notin (\dot{I}_{\bar{\beta}})^{H'_{\bar{\beta}}}$, then there exists $q \in H'_{\bar{\beta}}$ such that $q \Vdash \dot{X}_{\bar{\beta}} \notin \dot{I}_{\bar{\beta}}$. Since $m'_\alpha \restriction j_0(\bar{\beta}) \le j_0(q)$, we have $m'_\alpha \restriction \beta \Vdash j_0(\dot{R})_\beta$ is trivial and hence $m'_\alpha \restriction (\beta+1) \in j_0(Q)_{\beta+1}$.

Suppose $(\dot{X}_{\bar{\beta}})^{H'_{\bar{\beta}}} \in (\dot{I}_{\bar{\beta}})^{H_{\bar{\beta}}}$. So, there exists $p \in G$ and $q \in H'_{\bar{\beta}}$ such that $j(p)$ forces that for all $Q_{\bar{\beta}}$-generic filters $H''_{\bar{\beta}}$ over $V[\dot{G}]$ with $q \in H''_{\bar{\beta}}$ and all τ-weakly tight fast club subset C'' of κ over $V[\dot{G}][H''_{\bar{\beta}}]$ of order type ε, $\dot{m}'_\beta(H''_{\bar{\beta}}, C'') \Vdash_{j_0(Q_{\bar{\beta}})} \kappa \notin \dot{j}_{\bar{\beta}}(\dot{X})$. Since $G \subseteq \hat{G}$, we have $j(p) = p \in \hat{G}$. In $V[\hat{G}]$, $H'_{\bar{\beta}}$ is $Q_{\bar{\beta}}$-generic filter over $V[G]$ and $q \in H'_{\bar{\beta}}$. By definition, C' is a τ-weakly tight fast club subset of κ over $V[G][H'_{\bar{\beta}}]$ of order type ε. By the definition, we have $m'_\alpha \restriction \beta = m'_\alpha(H'_\alpha, C') \restriction \beta = m'_\beta(H'_\alpha \cap Q_{\bar{\beta}}, C')$. Thus, $m'_\alpha \restriction \beta \Vdash_{j_0(Q_{\bar{\beta}})} \kappa \notin \dot{j}_{\bar{\beta}}(\dot{X})$. So, $m'_\alpha \restriction \beta \Vdash_{j_0(Q_{\bar{\beta}})} m'_\alpha(\beta) = (\dot{D}_\beta)^{H'_{\bar{\beta}}} \cup \{\kappa\} \in P(\vec{C}, j(\kappa) \setminus \dot{j}_{\bar{\beta}}(\dot{X})) = j_0(\dot{R}_{\bar{\beta}}) = j_0(\dot{R})_\beta$. So, $m'_\alpha \restriction (\beta+1) \in j_0(Q)_{\beta+1}$. □

It is easy to see the remainder. □

Lemma 4.8. *In $V[G]$, Q_α forces that $\dot{I}_\alpha$ is non-trivial.*

Proof. Suppose that there exists $p \in G$ and $q \in Q_\alpha$ such that $q \Vdash$‘$\kappa \in \dot{I}_\alpha$’. Let $\hat{G}$ be $j(P)$-generic over V extending G. Then, by Lemma 4.2, there exists a Q_1-generic filter H'_1 with $q \restriction 1 \in H'_1$. By Lemma 4.4, there exists a τ-weakly tight fast club subset C' of κ over $V[G][H_1]$ of order type ε with $C' \in M[\hat{G}]$. By Lemma 4.6, there exists a Q_α-generic filter over $V[G]$ such that $H'_\alpha \in M[\hat{G}]$, $H'_\alpha \cap Q_1 = H'_1$, $q \in H'_\alpha$, and C' is a fast club subset of κ over $V[G][H'_\alpha]$.

Let $m'_\alpha = m'_\alpha(H'_\alpha, C')$. Let $\hat{H}_\alpha$ be a $j_0(Q_\alpha)$-generic filter over $V[\hat{G}]$ with $m'_\alpha \in \hat{H}_\alpha$. Since $q \Vdash$‘$\kappa \in \dot{I}_\alpha$’, we have $\kappa \notin j_\alpha(\kappa)$. This is a contradiction. □

Let $Q = Q_{\omega_2}$. We shall show that in $V[G]$, Q forces that $\mathrm{TCG}(\vec{C})$ is precipitous. In [6], interpreted in our terminology, they build a subposet Q^* of $j_0(Q)$ such that if H^* is a Q^*-generic filter over $V[\hat{G}]$, then H^* is $j_0(Q)$-generic over $M[\hat{G}]$ and ${j_0}^{-1}H^*$ is a Q-generic filter over $V[G]$. We shall use a different strategy, which is longer, but more intuitive and informative.

First, we shall prove some properties of I_α in $V[G][H_\alpha]$.

Lemma 4.9. *Let $\alpha < \omega_2$ and H_α be Q_α-generic over $V[G]$. Then, in $V[G][H_\alpha]$, $\check{\mathrm{TCG}}(\vec{C}) \subseteq I_\alpha$, and hence $\vec{C}$ is a tail club guessing sequence on ω_1.*

Proof. Suppose that there exists a club subset D of κ such that $X := \{ \delta \in \bar{D} \mid C_\delta \not\subseteq^* D \} \notin I_\alpha$. Let $p \in G$ and $q \in H_\alpha$ force this statement. Let $\dot{D}$ and $\dot{X}$ be $P * \dot{Q}_\alpha$-name for D and X.

Since $X \notin I_\alpha$, $j(p)$ does not force that for all Q_α-generic filter H'_α over $V[G]$ with $q \in H'_\alpha$ and all τ-weakly tight fast club subset C' of κ over $V[G][H'_\alpha]$ of order type ε, $m'_\alpha(H'_\alpha, C') \Vdash_{j_0(Q_\alpha)} \kappa \notin \dot{j}_\alpha(X')$ where $X' = \dot{X}^{H'_\alpha}$. So, there exists a $\hat{p} \le j(p)$ that forces there exist a Q_α-generic filter H'_α over $V[G]$ with $q \in H'_\alpha$ and a τ-weakly tight fast club subset C' of κ over $V[G][H'_\alpha]$ of order type ε such that $m'_\alpha(H'_\alpha, C') \nVdash_{j_0(Q_\alpha)} \kappa \notin \dot{j}_\alpha(X')$. Let $\hat{G}$ be $j(P)$-generic over V such that $\hat{p} \in \hat{G}$ and work in $V[\hat{G}]$. Let H'_α be a Q_α-generic filter over $V[G]$ with $q \in H'_\alpha$ and C' a τ-weakly tight fast club subset of κ over $V[G][H'_\alpha]$ of order type ε so that $m'_\alpha(H'_\alpha, C') \nVdash_{j_0(Q_\alpha)} \kappa \notin \dot{j}_\alpha(X')$. Let $D' = \dot{D}^{H'_\alpha}$.

On the other hand, since C' is a fast club subset of κ over $V[G][H'_\alpha]$, we have $C' \subseteq^* D'$. Notice $m'_\alpha(H'_\alpha, C) \Vdash_{j_0(Q_\alpha)} j_1(C)_\kappa = C' \wedge j_\alpha(D') \cap \kappa = D'$. So, $m'_\alpha(H'_\alpha, C) \Vdash_{j_0(Q_\alpha)} j_1(C)_\kappa \subseteq j_\alpha(D)$. Since $j(p) \in \hat{G}$ and $m'_\alpha(H'_\alpha, C') \le j_0(q)$, we have $m'_\alpha(H'_\alpha, C') \Vdash_{j_0(Q_\alpha)} \dot{j}_\alpha(X') = \{ \delta \in j_0(\bar{D}) \mid j_1(C)_\delta \not\subseteq^* \dot{j}_\alpha(D') \}$. So, $m'_\alpha(H'_\alpha, C') \Vdash_{j_0(Q_\alpha)} \kappa \notin j_\alpha(X)$. This is a contradiction. □

Let $\hat{G}$ be a $j(P)$-generic filter over V extending G and work in $M[\hat{G}]$. For every $\alpha < \omega_2$, let $m_\alpha \in \mathcal{B}(j_0(Q_\alpha))$ be the least upper bound of the set of all $m'_\alpha(H'_\alpha, C')$ such that H'_α is a Q_α-generic filter over $V[G]$ and C' is a τ-weakly tight fast club over $V[G][H'_\alpha]$ of order type ε. Since $\mathcal{P}(Q_\alpha)^{V[G]}$ and $\mathcal{P}(\kappa)^{V[G]}$ belong to $M[\hat{G}]$, m_α is definable in $M[\hat{G}]$. Let $\dot{m}_\alpha$ be a $j(P)/G$-name for m_α.

Lemma 4.10. *Let $\hat{G}$ be a $j(P)$-generic filter over V extending G, and let $\hat{H}_\alpha$ be a $j_0(Q_\alpha)$-generic filter over $M[\hat{G}]$ with $m_\alpha \in \hat{H}_\alpha$. Let $H'_\alpha = {j_0}^{-1}\hat{H}_\alpha$. Then, H'_α is a Q_α-generic filter over $V[G]$. In particular, we can define $j_\alpha : V[G][H'_\alpha] \to M[\hat{G}][\hat{H}_\alpha]$.*

Proof. Let $\hat{G}$, $\hat{H}_\alpha$, and H'_α be as in the assumption. Since $m_\alpha \in \hat{H}_\alpha$, there exist a Q_α-generic filter H''_α over $V[G]$ and a τ-weakly tight fast club subset C'' of κ over $V[G][H''_\alpha]$ of order type ε such that $m'_\alpha(H''_\alpha, C'') \in \hat{H}_\alpha$. By Lemma 4.7, we have ${j_0}^{-1}\hat{H}_\alpha = H''_\alpha$. So, $H'_\alpha = H''_\alpha$. Thus, H'_α is a Q_α-generic filter over $V[G]$. □

In $V[G]$, for each $\alpha < \kappa^+$, we shall define a function $\pi_\alpha : Q_\alpha * (\mathcal{P}(\omega_1)/\dot{I}_\alpha) \to \mathcal{B}((j(P)/G) * (\dot{j}_0(\dot{Q}_\alpha)/\dot{m}_\alpha))$. For every $\left\langle q, \dot{X} \right\rangle \in Q_\alpha * (\mathcal{P}(\omega_1)/\dot{I}_\alpha)$, let $\pi_\alpha\left(\left\langle q, \dot{X} \right\rangle\right)$ be the least upper bound of the set of all $\left\langle \hat{p}, \dot{\hat{q}} \right\rangle \in (j(P)/G) * (\dot{j}_0(\dot{Q}_\alpha)/\dot{m}_\alpha)$ such that $\hat{p}$ forces in $j(P)/G$ that there exist a Q_α-generic filter H'_α over $V[G]$ and a

τ-weakly tight fast club subset C' of κ over $V[G][H'_\alpha]$ of order type ε such that $q \in H'_\alpha$, $\dot{\hat{q}} \leq m'_\alpha(H'_\alpha, C')$, and $\dot{\hat{q}} \Vdash_{j_0(Q_\alpha)} \kappa \in \dot{j}_\alpha(\dot{X}^{H'_\alpha})$.

Lemma 4.11. *In $V[G]$, for every $\alpha < \kappa^+$, π_α is a dense embedding. In particular, $Q_\alpha * (\mathcal{P}(\omega_1)/\dot{I}_\alpha) \simeq (j(P)/G) * (\dot{j}_0(\dot{Q}_\alpha)/\dot{m}_\alpha)$*

Proof. It is easy to see that π_α is order-preserving. To see that π_α preserves incompatibility, let $\left\langle q_1, \dot{X}_1 \right\rangle$ and $\left\langle q_2, \dot{X}_2 \right\rangle$ be incompatible elements in $Q_\alpha * (\mathcal{P}(\kappa)/\dot{I}_\alpha)$. It follows that whenever q is a common extension of q_1 and q_2, $q \Vdash$ '$\dot{X}_1 \cap \dot{X}_2 \in \dot{I}_\alpha$'. Suppose that $\pi_\alpha\left(\left\langle q_1, \dot{X}_1 \right\rangle\right)$ and $\pi_\alpha\left(\left\langle q_2, \dot{X}_2 \right\rangle\right)$ are compatible in $\mathcal{B}((j(P)/G) * (\dot{j}_0(\dot{Q}_\alpha)/\dot{m}_\alpha))$.

Let $\hat{G}*\hat{H}_\alpha$ be a $(j(P)/G)*(\dot{j}_0(\dot{Q}_\alpha)/\dot{m}_\alpha)$-generic filter over $V[G]$ with $\pi_\alpha\left(\left\langle q_1, \dot{X}_1 \right\rangle\right), \pi_\alpha\left(\left\langle q_2, \dot{X}_2 \right\rangle\right) \in \hat{G}*\hat{H}_\alpha$. For $i = 1, 2$, since $\pi_\alpha\left(\left\langle q_i, \dot{X}_i \right\rangle\right) \in \hat{G}*\hat{H}_\alpha$, in $V[\hat{G}]$, there exist a Q_α-generic filter $H'_{i,\alpha}$ over $V[G]$ and a τ-weakly tight fast club subset C'_i of κ over $V[G][H'_{i,\alpha}]$ of order type ε such that $q_i \in H'_{i,\alpha}$, $m'_\alpha(H'_{i,\alpha}, C'_i) \in \hat{H}_\alpha$, and in $M[\hat{G}][\hat{H}_\alpha]$, $\kappa \in j_\alpha(\dot{X}_i^{H'_{i,\alpha}})$. By Lemma 4.7, since $m_\alpha \in \hat{H}_\alpha$, we have $H'_{1,\alpha} = H'_{2,\alpha} = {j_0}^{-1}\hat{H}_\alpha$ and $C'_1 = C'_2 = j_1(C)_\kappa$. So, let $H'_\alpha = H'_{1,\alpha}$ and $C' = C'_1$. For each $i = 1, 2$, let $X_i = (\dot{X}_i)^{H'_\alpha}$. So, in $M[\hat{G}][\hat{H}_\alpha]$, $\kappa \in j_\alpha(X_1 \cap X_2)$.

Since $q \Vdash \dot{X}_1 \cap \dot{X}_2 \in \dot{I}_\alpha$, $m_\alpha(H'_\alpha, C') \Vdash_{j_0(Q_\alpha)} \kappa \notin \dot{j}_\alpha(X_1 \cap X_2)$. Since $m'_\alpha(H'_\alpha, C') \in \hat{H}_\alpha$, we have $\kappa \notin j_\alpha(X_1 \cap X_2)$. This is a contradiction. □

Lemma 4.12. *Let $\alpha < \kappa$. In $V[G]$, let H_α be Q_α-generic over $V[G]$. Then, in $V[G][H_\alpha]$, I_α is precipitous.*

Proof. Work in $V[G]$. Suppose that Q_α does not forces that $\dot{I}_\alpha$ is precipitous. Then, there exist $q_0 \in Q_\alpha$ and a Q_α-name $\dot{Y}_0$ for an element of $\mathcal{P}(\kappa)/\dot{I}_\alpha$ such that $\left\langle q_0, \dot{Y}_0 \right\rangle \Vdash$'$(V[G][\dot{H}_\alpha])^\kappa/\dot{U}_\alpha$ is not well-founded'. Here $\dot{U}_\alpha$ is a $Q_\alpha*(\mathcal{P}(\kappa)/\dot{I}_\alpha)$-name for a $\mathcal{P}(\kappa)/\dot{I}_\alpha$-generic filter over $V[G][\dot{H}_\alpha]$. Let $\hat{G}*\hat{H}_\alpha$ be a $(j(P)/G)*(\dot{j}_0(\dot{Q}_\alpha)/\dot{m}_\alpha)$-filter over $V[G]$ with $\pi_\alpha\left(\left\langle q_0, \dot{Y}_0 \right\rangle\right) \in \hat{G} * \hat{H}_\alpha$. Let $H_\alpha * U_\alpha = {\pi_\alpha}^{-1}(\hat{G} * \hat{H}_\alpha)$. Since π_α is a dense embedding, $H_\alpha * U_\alpha$ is a $Q_\alpha*(\mathcal{P}(\kappa)/\dot{I}_\alpha)$-generic filter over $V[G]$. Since $m_\alpha \in \hat{H}_\alpha$, there exist a Q_α-generic filter H'_α over $V[G]$ and a τ-weakly tight fast club subset C' of κ over $V[G][H'_\alpha]$ such that $m'_\alpha(H'_\alpha, C') \in \hat{H}_\alpha$. So, $j_1(C)_\kappa = C'$ and ${j_0}^{-1}\hat{H}_\alpha = H'_\alpha$.

Claim 4.12.1. $H_\alpha = H'_\alpha$.

Proof. First, we shall show j_0“$H_\alpha \subseteq \hat{H}_\alpha$. Let $q \in H_\alpha$. Then, $\pi_\alpha(\langle q, \kappa \rangle) \in \hat{G} * \hat{H}_\alpha$. So, there exist a Q_α-generic filter H'_α over $V[G]$ and a τ-weakly tight fast club subset C' of κ over $V[G][H'_\alpha]$ such that $q \in H'_\alpha$, $m'_\alpha(H'_\alpha, C') \in \hat{H}_\alpha$, and in $V[\hat{G}][\hat{H}_\alpha]$, $\kappa \in j_\alpha(\dot{Y}^{H'_\alpha})$. But by the definition, since $q \in H'_\alpha$, we have $m'_\alpha(H'_\alpha, C') \leq j_0(q)$. So, $j_0(q) \in \hat{H}_\alpha$.

Since j_0 is injective, it implies $H_\alpha = {j_0}^{-1}\hat{H}_\alpha = H'_\alpha$. □

Claim 4.12.2. For every $Y \in U_\alpha$, $\kappa \in j_\alpha(Y)$.

Proof. Let $\dot{Y}$ be a Q_α-name for Y. Then, $\pi_\alpha\left(\left\langle \mathbb{1}_{Q_\alpha}, \dot{Y} \right\rangle\right) \in \hat{G} * \hat{H}_\alpha$. So, there exist a Q_α-generic filter H''_α over $V[G]$ and a τ-weakly tight fast club subset C''

of κ over $V[G][H''_\alpha]$ of order type ε such that $m'_\alpha(H''_\alpha, C') \in \hat{H}_\alpha$ and in $V[\hat{G}][\hat{H}_\alpha]$, $\kappa \in j_\alpha(\dot{Y}^{H''_\alpha})$. By Lemma 4.7, $H''_\alpha = {j_0}^{-1}\hat{H}_\alpha = H_\alpha$. So, $\kappa \in j_\alpha(\dot{Y}^{H_\alpha}) = j_\alpha(Y)$. □

Since $\left\langle q_0, \dot{Y}_0 \right\rangle \in H_\alpha * \dot{U}_\alpha$, $(V[G][H_\alpha])^\kappa/U_\alpha$ is not well-founded.

Let $U_0 = U_\alpha \cap V[G]$. As in [6], we can show that $(V[G])^\kappa/U_0 \simeq M[\hat{G}]$ and j_0 coincides with the generic elementary embedding induced by U_0.

Define a function $\sigma_\alpha : (V[G][H_\alpha])^\kappa/U_\alpha \to M[\hat{G}][\hat{H}_\alpha]$ as follows. Let $f : \kappa \to V[G][H_\alpha]$ be a function lying in $V[G][H_\alpha]$. Let $\sigma_\alpha([f]_{U_\alpha}) = j_\alpha(f)(\kappa)$. First, we shall show that σ_α is well-defined. Suppose that $[f]_{U_\alpha} = [g]_{U_\alpha}$. Let $Y = \{\, \xi < \kappa \mid f(\xi) = g(\xi) \,\}$. Then, since $Y \in U_\alpha$, we have $\kappa \in j_\alpha(Y)$. So, $j_\alpha(f)(\kappa) = j_\alpha(g)(\kappa)$ and hence $\sigma_\alpha([f]_{U_\alpha}) = \sigma_\alpha([g]_{U_\alpha})$. By a similar argument, we can show that $[f]_{U_\alpha} \in_{U_\alpha} [g]_{U_\alpha}$ if and only if $\sigma_\alpha([f]_{U_\alpha}) \in \sigma_\alpha([g]_{U_\alpha})$.

To see that σ_α is onto, let $x \in M[\hat{G}][\hat{H}_\alpha]$. Then, there exists a $j_0(Q_\alpha)$-name $\dot{x} \in M[\hat{G}]$ such that $x = \dot{x}^{\hat{H}_\alpha}$. Since $(V[G])^\kappa/U_0 \simeq M[\hat{G}]$, there exists a function $f : \kappa \to V[G]$ lying in $V[G]$ such that for every $\xi < \kappa$, $f(\xi)$ is a Q_α-name and $j_0(f)(\kappa) = \dot{x}$. Define a function $g : \kappa \to V[G][H_\alpha]$ by $g(\xi) = {f(\xi)}^{H_\alpha}$. Then, $\sigma_\alpha([g]_{U_\alpha}) = j_\alpha(g)(\kappa) = j_0(f)(\kappa)^{\hat{H}_\alpha} = \dot{x}^{\hat{H}_\alpha} = x$.

Therefore, $\sigma_\alpha : (V[G][H_\alpha])^\kappa/U_\alpha \to M[\hat{G}][\hat{H}_\alpha]$ is an isomorphism. Since $M[\hat{G}][\hat{H}_\alpha]$ is well-founded, so is $(V[G][H_\alpha])^\kappa/U_\alpha$. This is a contradiction. □

Now, we shall show that for all $\beta < \alpha < \omega_2$, $Q_\beta*(\mathcal{P}(\kappa)/\dot{I}_\beta)$ is regularly embedded into $Q_\alpha * (\mathcal{P}(\kappa)/\dot{I}_\alpha)$

Lemma 4.13. *In $V[G]$, for every $\beta < \alpha < \kappa^+$, $j(P)/G$ forces that $\dot{m}_\beta = \dot{m}_\alpha \restriction j(\beta)$.*

Proof. Let $\hat{G}$ be $j(P)$-generic over V extending G and work in $V[\hat{G}]$.

First, We shall show that $m_\alpha \restriction j(\beta) \le m_\beta$. Suppose not, i.e. $m_\alpha \restriction j(\beta) \not\le m_\beta$. Let $q \in j_0(Q_\beta)$ be so that $q \le m_\alpha \restriction j(\beta)$ and q is incompatible with m_β. By the definition of m_α, there exist a Q_α-generic filter H'_α over $V[G]$ and a τ-weakly tight fast club subset C' of κ over $V[G]$ of order type ε such that q is compatible with $m'_\alpha(H'_\alpha, C') \restriction j(\beta)$. Let q' be a common extension of q and $m'_\alpha(H'_\alpha, C') \restriction j(\beta)$. Notice that $m'_\alpha(H'_\alpha, C') \restriction j(\beta) = m'_\beta(H'_\alpha \cap Q_\beta, C') \le m_\beta$. So, $q' \le m_\beta$. This is a contradiction.

Conversely, we shall show that $m_\beta \le m_\alpha \restriction j(\beta)$. Suppose not, i.e. $m_\beta \not\le m_\alpha \restriction j(\beta)$. Let $q \in j_0(Q_\beta)$ be so that $q \le m_\beta$ and q is incompatible with $m_\alpha \restriction j(\beta)$. By the definition of m_β, there exists a Q_β-generic filter H'_β over $V[G]$ and a τ-weakly tight fast club subset C' of κ over $V[G]$ of order type ε such that q is compatible with $m'_\beta(H'_\beta, C')$. Let q' be a common extension of q and $m'_\beta(H'_\beta, C')$. By Lemma 4.6, there exists a Q_α-generic filter H'_α over $V[G]$ such that $H'_\alpha \in M[\hat{G}]$, $H'_\beta = H'_\alpha \cap Q_\beta$, and C' is a fast club subset of κ over $V[G][H'_\alpha]$. Then, $m'_\beta(H'_\beta, C') = m'_\alpha(H'_\alpha, C') \restriction j(\beta)$. Thus, $q' \le m'_\alpha(H'_\alpha, C') \restriction j(\beta) \le m_\alpha \restriction j(\beta)$. This is a contradiction to the assumption that q is incompatible with $m_\alpha \restriction j(\beta)$. □

Suppose that $\hat{G}$ is $j(P)$-generic over V extending G and $\hat{H}_\alpha$ is $j_0(Q_\alpha)$-generic over $M[\hat{G}]$ with $m_\alpha \in \hat{H}_\alpha$. For every $\beta < \alpha$, define $\hat{H}_\beta = \hat{H}_\alpha \cap j_0(Q_\beta)$ and $H_\beta = {j_0}^{-1}\hat{H}_\beta$. Since $m_\beta = m_\alpha \restriction j(\beta) \in \hat{H}_\beta$, we can define $j_\beta : V[G][H_\beta] \to M[\hat{G}][\hat{H}_\beta]$ as we do for j_α. It is easy to see that j_β is an elementary embedding and $j_\beta = j_\alpha \restriction V[G][H_\beta]$.

Lemma 4.14. *Let $\beta < \alpha < \kappa^+$. Suppose that H_α is Q_α-generic over $V[G]$. Then, in $V[G][H_\alpha]$, $I_\beta = I_\alpha \cap V[G][H_\alpha \cap Q_\beta]$.*

Proof. Let $H_\beta = H_\alpha \cap Q_\beta$.

Let $X \in I_\beta$. Let $\dot{X}$ be a Q_β-name for X. Since $X \in I_\beta$, there exists $p \in G$ and $q \in H_\beta$ such that $j(p)$ forces that for all Q_β-generic filters H'_β over $V[G]$ with $q \in H'_\beta$ and all τ-weakly tight fast club subsets C' of κ over $V[G][H'_\beta]$ of order type ε, $\dot{m}'_\beta(H'_\beta, C') \Vdash_{j_0(Q_\beta)} \kappa \notin \dot{j}_\beta(X')$ where $X' = \dot{X}^{H'_\beta}$. To show that $X \in I_\alpha \cap V[G][H_\beta]$, suppose otherwise. Then, there exists $p' \in G$ and $q' \in H_\alpha$ such that $\langle p', q' \rangle \Vdash \dot{X} \notin \dot{I}_\alpha$. Without loss of generality, we may assume $p' \leq p$ and $q' \leq q$. So, there exists $\hat{p} \leq j(p')$ that forces that there exist a Q_α-generic filter H'_α over $V[G]$ with $q' \in H'_\alpha$ and a τ-weakly tight fast club subset C' of κ over $V[G][H'_\alpha]$ of order type ε such that $\dot{m}'_\alpha(H'_\alpha, C') \not\Vdash_{j_0(Q_\alpha)} \kappa \notin \dot{j}_\alpha(X')$ where $X' = \dot{X}^{H'_\alpha \cap Q_\beta}$.

Let $\hat{G}$ be a $j(P)$-generic filter over $V[G]$ extending G with $\hat{p} \in \hat{G}$. So, in $V[\hat{G}]$, there exist a Q_α-generic filter H'_α over $V[G]$ with $q' \in H'_\alpha$ and a τ-weakly tight fast club subset C' of κ over $V[G][H'_\alpha]$ of order type ε such that $m'_\alpha(H'_\alpha, C') \not\Vdash_{j_0(Q_\alpha)} \kappa \notin \dot{j}_\alpha(X')$ where $X' = \dot{X}^{H'_\alpha \cap Q_\beta}$. Let $H'_\beta = H'_\alpha \cap Q_\beta$. We have $m'_\beta(H'_\beta, C') \Vdash_{j_0(Q_\beta)} \kappa \notin \dot{j}_\beta(X')$. Since $m'_\alpha(H'_\alpha, C') \restriction j(\beta) = m'_\beta(H'_\beta, C')$ and $m'_\alpha(H'_\alpha, C') \Vdash \dot{j}_\alpha \restriction V[G][H'_\alpha] = \dot{j}_\beta$, we have $m'_\alpha(H'_\alpha, C') \Vdash_{j_0(Q_\alpha)} \kappa \notin \dot{j}_\alpha(X')$. This is a contradiction.

Now, we shall show that $I_\alpha \cap V[G][H_\alpha \cap Q_\beta] \subseteq I_\beta$. Suppose otherwise. Let $X \in (I_\alpha \cap V[G][H_\alpha \cap Q_\beta]) \setminus I_\beta$ and $\dot{X}$ a Q_β-name for X. Then, there exists $q_1 \in H_\alpha \cap Q_\beta$ such that $q_1 \Vdash \dot{X} \notin \dot{I}_\beta$. Since $X \in I_\alpha$, there exists $p_0 \in G$ and $q_0 \in H_\alpha$ such that $j(p_0)$ forces that for all Q_α-generic filters H'_α over $V[\dot{G}]$ with $q_0 \in H'_\alpha$ and for all τ-weakly tight fast club subset C of κ over $V[\dot{G}]$ of order type ε, $\dot{m}'_\alpha(H'_\alpha, C) \Vdash_{j_0(Q_\alpha)} \kappa \notin \dot{j}_\alpha(X')$ where $X' = \dot{X}^{H_\alpha \cap Q_\beta}$. Without loss of generality, we may assume $q_0 \restriction \beta \leq q_1$. Work in $V[G]$. Let $\hat{G}$ be a $j(P)$-generic filter over V extending G. Then, $j(p_0) = p_0 \in \hat{G}$. Let $H'_\beta \in M[\hat{G}]$ be a Q_β-generic filter over $V[G]$ with $q_0 \restriction \beta \in H'_\beta$ and C' a τ-weakly tight fast club subset of κ over $V[G]$ of order type ε. By Lemma 4.6, there exists a Q_α-generic filter H'_α over $V[G]$ such that $H'_\alpha \in M[\hat{G}]$, $H'_\beta = H'_\alpha \cap Q_\beta$, $q \in H'_\alpha$, and C' is a fast club subset of κ over $V[G][H'_\alpha]$. So, $m'_\alpha(H'_\alpha, C') \Vdash_{j_0(Q_\alpha)} \kappa \notin \dot{j}_\alpha(X')$. Since $m'_\beta(H'_\beta, C') = m'_\alpha(H'_\alpha, C') \restriction j(\beta)$ and $m'_\alpha(H'_\alpha, C') \Vdash \dot{j}_\alpha(X') = \dot{j}_\beta(X')$, we have $m'_\beta(H'_\beta, C') \Vdash_{j_0(Q_\beta)} \kappa \notin \dot{j}_\beta(X')$. So, $q_0 \restriction \beta \Vdash X' \in \dot{I}_\beta$. This is a contradiction since $q_0 \restriction \beta \leq q_1$ and $q_1 \Vdash \dot{X} \notin \dot{I}_\beta$.

□

Let $\beta < \alpha < \omega_2$. Since $Q_\beta = Q_\alpha \restriction \beta$, every element of Q_β can be regarded as an element of Q_α. Let $\dot{X}$ be a Q_β-name for an element of $\mathcal{P}(\omega_1)/\dot{I}_\beta$. Namely, $\mathbb{1}_{Q_\beta} \Vdash \dot{X} \subseteq \omega_1 \wedge \dot{X} \notin \dot{I}_\beta$. Then, $\dot{X}$ can be regarded as a Q_α-name. Moreover, by Lemma 4.14, $\mathbb{1}_{Q_\alpha} \Vdash \dot{X} \notin \dot{I}_\alpha$. Thus, $\dot{X}$ can be regarded as a Q_α-name for an element of $\mathcal{P}(\omega_1)/\dot{I}_\alpha$. So, we have $Q_\beta * (\mathcal{P}(\omega_1)/\dot{I}_\beta) \subseteq Q_\alpha * (\mathcal{P}(\omega_1)/\dot{I}_\alpha)$.

Lemma 4.15. *In $V[G]$, for every $\beta < \alpha < \omega_2$, $Q_\beta * (\mathcal{P}(\omega_1)/\dot{I}_\beta)$ is regularly embedded into $Q_\alpha * (\mathcal{P}(\omega_1)/\dot{I}_\alpha)$ by the inclusion map.*

Proof. It is easy to observe that for every $\left\langle q, \dot{X}\right\rangle \in Q_\beta * (\mathcal{P}(\omega_1)/\dot{I}_\beta)$, $\pi_\beta\left(\left\langle q, \dot{X}\right\rangle\right) = \pi_\alpha\left(\left\langle q, \dot{X}\right\rangle\right)$. So, the following diagram commutes.

$$\begin{array}{ccc} Q_\beta * (\mathcal{P}(\omega_1)/\dot{I}_\beta) & \xrightarrow{\pi_\beta} & (j(P)/G) * (\dot{j}_0(\dot{Q}_\beta)/\dot{m}_\beta) \\ {\scriptstyle\text{incl}}\downarrow & & {\scriptstyle\text{incl}}\downarrow \\ Q_\alpha * (\mathcal{P}(\omega_1)/\dot{I}_\alpha) & \xrightarrow{\pi_\alpha} & (j(P)/G) * (\dot{j}_0(\dot{Q}_\alpha)/\dot{m}_\alpha) \end{array}$$

Therefore, the conclusion holds. □

When H is a Q-generic filter over $V[G]$, let $I = \bigcup_{\alpha<\omega_2} I_\alpha$. Let $\dot{I}$ be the Q-name for I.

Lemma 4.16. *In $V[G]$, $Q*(\mathcal{P}(\omega_1)/\dot{I})$ is the direct limit of $\left\langle Q_\alpha * (\mathcal{P}(\omega_1)/\dot{I}_\alpha) \,\middle|\, \alpha < \omega_2\right\rangle$.*

Proof. By definition, for every $\alpha < \omega_2$, $Q = Q_{\omega_2}$ is a direct limit of $\langle Q_\alpha \mid \alpha < \omega_2\rangle$. Let H be a Q-generic filter over $V[G]$. Then, by Lemma 4.14, for every $\alpha < \omega_2$, $I_\alpha = I \cap V[G][H \cap Q_\alpha]$. By Lemma 4.15, $Q * (\mathcal{P}(\omega_1)/\dot{I})$ is the direct limit of $\left\langle Q_\alpha * (\mathcal{P}(\omega_1)/\dot{I}_\alpha) \,\middle|\, \alpha < \omega_2\right\rangle$. □

Lemma 4.17. *Let H be a Q-generic filter over $V[G]$. Then, in $V[G][H]$, $I = \mathrm{T\breve{C}G}(\vec{C})$.*

Proof. By Lemma 4.9, $\mathrm{T\breve{C}G}(\vec{C}) \subseteq I$. To show $I \subseteq \mathrm{T\breve{C}G}(\vec{C})$, let $X \in I$. Then, there exists $\alpha < \omega_2$ such that $X = X_\alpha$. Since $X \in I$, by Lemma 4.14, we have $X_\alpha \in I_\alpha$. So, $R_\alpha = P(\vec{C}, \kappa \setminus X_\alpha)$. Thus, $\kappa \setminus X_\alpha \in \mathrm{TCG}(\vec{C})$, and hence $X_\alpha \in \mathrm{TC\breve{G}}(\vec{C})$. □

Lemma 4.18. *Let $G * H$ be $P * \dot{Q}$-generic over V. Then, in $V[G][H]$, $\mathrm{TCG}(\vec{C})$ is precipitous.*

Proof. Work in $V[G]$. By Lemma 4.17, Q forces that $\mathrm{T\breve{C}G}(\dot{\vec{C}}) = I$. We shall show that Q forces that I is precipitous. Suppose not. Then, there exist $q_0 \in Q$ and a Q-name $\dot{Y}_0$ for an element of $\mathcal{P}(\omega_1)/\dot{I}$ such that $\left\langle q_0, \dot{Y}_0\right\rangle \Vdash_{Q*(\mathcal{P}(\omega_1)/\dot{I})}$ '$(V[G][\dot{H}])^{\omega_1}/\dot{U}$ is not well-founded'. Here, $\dot{U}$ is a $Q * (\mathcal{P}(\omega_1)/\dot{I})$-name for a $\mathcal{P}(\omega_1)/\dot{I}$-generic filter over $V[G][\dot{H}]$. Then, there exists a sequence $\left\langle \dot{f}_n \mid n < \omega\right\rangle$ of $Q*(\mathcal{P}(\kappa)/\dot{I})$-names for functions from ω_1 into ON such that $\left\langle q_0, \dot{Y}_0\right\rangle \Vdash$ 'for all $n < \omega$, $[\dot{f}_{n+1}]_{\dot{U}} <_{\dot{U}} [\dot{f}_n]_{\dot{U}}$'. Without loss of generality, we may assume that there exists $\alpha < \omega_2$ such that $q_0 \in Q_\alpha$, $\dot{Y}_0$ is a Q_α-name, and for each $n < \omega$, $\dot{f}_n$ is Q_α-name. By Lemma 4.16, $Q_\alpha * (\mathcal{P}(\omega_1)/\dot{I}_\alpha)$ is regularly embedded into $Q * \mathcal{P}(\omega_1)/\dot{I}$. So, it means that $\left\langle q_0, \dot{Y}_0\right\rangle \Vdash_{Q_\alpha*(\mathcal{P}(\omega_1)/\dot{I}_\alpha)}$ 'for all $n < \omega$, $[\dot{f}_{n+1}]_{\dot{U}_\alpha} <_{\dot{U}_\alpha} [\dot{f}_n]_{\dot{U}_\alpha}$'. It implies that q_0 forces that $\dot{I}_\alpha$ is not precipitous. This is a contradiction to Lemma 4.12. □

5. No restriction of NS_{ω_1} is precipitous in $V[G][H]$

In this section and the next one, suppose P, Q, Q_α, R_α, $\left\langle C_\delta \mid \delta \in \bar{D}\right\rangle$, I, and I_α are in the previous section.

For every $\alpha < \kappa^+$ and $\zeta < \kappa$, let $\dot{\mathcal{D}}_{\alpha,\zeta}$ be a P-name for $\mathcal{D}_{\alpha,\zeta}$. Define $\bar{\mathcal{D}}_\alpha$ be the set of all sets of the form $\langle p, \dot{q}\rangle$ such that $p \in P$, $p \Vdash \dot{q} \in \dot{\mathcal{D}}_{\alpha,0}$, and p decides $\dot{q}$.

If $p \in \mathrm{Coll}(\omega, \leq \zeta)$ and $p \Vdash \mathrm{ht}(\dot{q}) = \zeta$, we say that $\langle p, \dot{q}\rangle$ has height ζ and write $\mathrm{ht}(\langle p, \dot{q}\rangle) = \zeta$. Clearly, $\bar{\mathcal{D}}_\alpha$ is dense in $P * \dot{Q}_\alpha$. It is easy to see that $\bar{\mathcal{D}}_\alpha \in H(\kappa^+)^V$ and hence $\bar{\mathcal{D}}_\alpha \in M$.

Similarly, let $\alpha < \beta < \kappa^+$. Let $G_{\bar{\mathcal{D}}_\alpha}$ be $\bar{\mathcal{D}}_\alpha$-generic over V. In $V[G_{\bar{\mathcal{D}}_\alpha}]$, define $\bar{\mathcal{D}}_{\alpha,\beta} = \{ (\dot{q} \restriction [\alpha, \beta))^{G_{\bar{\mathcal{D}}_\alpha}} \mid \langle p, \dot{q}\rangle \in \bar{\mathcal{D}}_\beta \}$. When $\mathrm{ht}(\langle p, \dot{q}\rangle) = \zeta$, we say that $(\dot{q} \restriction [\alpha, \beta))^{G_{\bar{\mathcal{D}}_\alpha}} = \zeta$ and write $\mathrm{ht}((\dot{q} \restriction [\alpha, \beta))^{G_{\bar{\mathcal{D}}_\alpha}}) = \zeta$. Let $\dot{\bar{\mathcal{D}}}_{\alpha,\beta}$ be a $\bar{\mathcal{D}}_\alpha$-name for $\bar{\mathcal{D}}_{\alpha,\beta}$. Clearly, we have $\bar{\mathcal{D}}_\beta$ is forcing equivalent to $\bar{\mathcal{D}}_\alpha * \dot{\bar{\mathcal{D}}}_{\alpha,\beta}$.

Lemma 5.1. *Suppose $G * H$ is a $P * \dot{Q}$-generic filter. Then, in $V[G][H]$, for every stationary subset S of ω_1, there exists a stationary subset S' of S such that $S' \in \mathrm{T\check{C}G}(\vec{C})$.*

Proof. Work in $V[G]$. Notice that Q is a countable support iteration of length ω_2. So, it adds a Cohen subset of ω_1 unboundedly many times. But it is easy to see that if S is a stationary subset of ω_1, after adding a Cohen subset of ω_1, there exists a stationary subset S' of S such that $S' \in \mathrm{T\check{C}G}(\vec{C})$. Therefore, the conclusion holds. □

Lemma 5.2. *Assume $V = L[U]$. Suppose $G * H$ is a $P * \dot{Q}$-generic filter. For every stationary subset S of ω_1, $\mathrm{NS}_{\omega_1} \restriction S$ is not precipitous in $V[G][H]$.*

Proof. Suppose that $\dot{S}$ is a $P * \dot{Q}$-name for a stationary subset of κ, and $\langle p_0, \dot{q}_0\rangle$ forces that $\mathrm{NS}_{\omega_1} \restriction \dot{S}$ is precipitous. By Lemma 5.1, without loss of generality, we may assume that there exists $\alpha < \kappa^+$ such that $\langle p_0, \dot{q}_0\rangle \in \bar{\mathcal{D}}_\alpha$, $\dot{S}$ is a $\bar{\mathcal{D}}_\alpha$-name, and $\langle p_0, \dot{q}_0\rangle \Vdash \dot{X}_\alpha \in \dot{I}_\alpha$.

Let f be a function in V with domain κ such that for every $\gamma < \kappa$, $f(\gamma)$ is the set of all $s \in \bar{\mathcal{D}}_{\alpha+1}$ that forces $(\nu_{\bar{\mathcal{D}}_{\alpha+1}}(\gamma))^{-1}\left(\dot{G}_{\bar{\mathcal{D}}_{\alpha+1}}\right)$ is not $\rho_{\bar{\mathcal{D}}_{\alpha+1}}(\gamma)$-generic over V. Let $\dot{S}'$ be a $\bar{\mathcal{D}}_{\alpha+1}$-name for a subset of $\dot{S}$ such that for every $\gamma < \kappa$, $\langle p, \dot{q}\rangle \Vdash \gamma \in \dot{S}'$ if and only if $\langle p, \dot{q}\rangle \in f(\gamma)$.

Claim 5.2.1. Let $G * H$ be $P * \dot{Q}$-generic over V. Let J be a κ-complete normal precipitous ideal on κ in $V[G][H]$. Then, $(\dot{S}')^{G*H} \in J$.

Proof. Work in V. Let $\dot{J}$ be $P * \dot{Q}$-name for J. Let $\langle p_1, \dot{q}_1\rangle \in P * \dot{Q}$ be so that $\langle p_1, \dot{q}_1\rangle$ forces that $\dot{J}$ is a κ-complete normal precipitous ideal on κ. By way of contradiction, suppose $\langle p_1, \dot{q}_1\rangle \Vdash \dot{S}' \notin \dot{J}$.

Let $G * H$ be $P * \dot{Q}$-generic over V with $\langle p_1, \dot{q}_1\rangle \in G * H$. Let $J = \dot{J}^{G*H}$ and $S' = (\dot{S}')^{G*H}$. Let W be $\mathcal{P}(\kappa)/J$-generic over $V[G][H]$ with $S' \in W$ and $k : V[G][H] \to M'$ the induced generic elementary embedding. By Lemma 3.2, we have $M' = L[k(U)][k(G)][k(H)]$. Then, $k(G * H)$ is $k(P * \dot{Q})$-generic over $L[k(U)]$. Since $S' \in W$, we have $\kappa \in k(S')$. Since $S' = (\dot{S}')^{G*H}$, we have $k(S') = (k(\dot{S}'))^{k(G*H)}$. Since $\kappa \in k(S') = (k(\dot{S}'))^{k(G*H)}$, there exists $\hat{s} \in k(G * H)$ such that $\hat{s} \Vdash \kappa \in k(\dot{S}')$. Since $\dot{S}'$ is a $\bar{\mathcal{D}}_{\alpha+1}$-name, $k(\dot{S}')$ is a $k(\bar{\mathcal{D}}_{\alpha+1})$-name. So, we may assume that $\hat{s} \in k(\bar{\mathcal{D}}_{\alpha+1})$. Since $\hat{s} \in k(\bar{\mathcal{D}}_{\alpha+1})$, there exists a function $g : \kappa \to \bar{\mathcal{D}}_{\alpha+1}$ lying in $V[G][H]$ such that $[g]_W = \hat{s}$.

Let $G_{\bar{\mathcal{D}}_{\alpha+1}} = (G * H) \cap \bar{\mathcal{D}}_{\alpha+1}$ and $G_{k(\bar{\mathcal{D}}_{\alpha+1})} = (\dot{G}_{k(\mathcal{D}_{\alpha+1})})^{k(G*H)}$. Then, $G_{k(\bar{\mathcal{D}}_{\alpha+1})} = k(G_{\bar{\mathcal{D}}_{\alpha+1}})$.

Then,

$$\begin{aligned}
&\hat{s} \Vdash \kappa \in k(\dot{S}') \\
&\iff [g]_W \Vdash \kappa \in k(\dot{S}') \\
&\iff \{\, \gamma < \kappa \mid g(\gamma) \Vdash \gamma \in \dot{S}' \,\} \in W \\
&\iff \{\, \gamma < \kappa \mid g(\gamma) \in f(\gamma) \,\} \in W \\
&\iff \{\gamma < \kappa \mid g(\gamma) \text{ forces } (\nu_{\bar{\mathcal{D}}_{\alpha+1}}(\gamma))^{-1} \left(\dot{G}_{\bar{\mathcal{D}}_{\alpha+1}}\right) \text{ is not} \\
&\qquad \rho_{\bar{\mathcal{D}}_{\alpha+1}}(\gamma)\text{-generic over } V\} \in W \\
&\iff [g]_W \text{ forces } k^{-1}\dot{G}_{k(\mathcal{D}_{\alpha+1})} \text{ is not } \bar{\mathcal{D}}_{\alpha+1}\text{-generic over } L[k(U)] \\
&\iff \hat{s} \text{ forces } k^{-1}\dot{G}_{k(\mathcal{D}_{\alpha+1})} \text{ is not } \bar{\mathcal{D}}_{\alpha+1}\text{-generic over } L[k(U)]
\end{aligned}$$

Thus, $k^{-1}G_{k(\bar{\mathcal{D}}_{\alpha+1})}$ is not $\bar{\mathcal{D}}_{\alpha+1}$-generic over $L[k(U)]$. However, $k^{-1}G_{k(\bar{\mathcal{D}}_{\alpha+1})} = G_{\bar{\mathcal{D}}_{\alpha+1}}$, which is $\bar{\mathcal{D}}_{\alpha+1}$-generic over V. By Theorem 3.1, $L[k(U)]$ is an iterated ultrapower of $V = L[U]$. Thus, we have $L[k(U)] \subseteq V$. So, $k^{-1}G_{k(\bar{\mathcal{D}}_{\alpha+1})}$ is $\bar{\mathcal{D}}_{\alpha+1}$-generic over $L[k(U)]$. This is a contradiction. $\square$

Claim 5.2.2. $\langle p_0, \dot{q}_0\rangle \in P * \dot{Q}$ forces that $\dot{S}'$ is stationary.

Proof. Suppose not. Then, there exists $\langle p_1, \dot{q}_1\rangle \in P * \dot{Q}$ and a $P * \dot{Q}$-name $\dot{D}$ for a club subset of κ such that $\langle p_1, \dot{q}_1\rangle \leq \langle p_0, \dot{q}_0\rangle$ and $\langle p_1, \dot{q}_1\rangle \Vdash \dot{S}' \cap \dot{D} = \varnothing$. Without loss of generality, we may assume that for some $\beta < \kappa^+$, $\langle p_1, \dot{q}_1\rangle \in \bar{\mathcal{D}}_\beta$ and $\dot{D}$ is a $\bar{\mathcal{D}}_\beta$-name. Without loss of generality, we may assume $p_1 \Vdash \alpha \in \operatorname{supp}(\dot{q}_1)$.

Let $G * H$ be $P * \dot{Q}$-generic over V with $\langle p_1, \dot{q}_1\rangle \in G * H$. Work in $V[G][H]$. Let $S = \dot{S}^{G*H}$ and $S' = (\dot{S}')^{G*H}$. By assumption, $\mathrm{NS}_\kappa \restriction S$ is precipitous. Let W be $\mathcal{P}(\kappa)/\mathrm{NS}_\kappa \restriction S$-generic over $V[G][H]$ and $k : V[G][H] \to L[k(U)][k(G)][k(H)]$ the induced generic elementary embedding. Since $S \in W$, we have $\kappa \in k(S)$.

Let $G_{\bar{\mathcal{D}}_\alpha} = (G * H) \cap \bar{\mathcal{D}}_\alpha$. Then, $G_{\bar{\mathcal{D}}_\alpha}$ is $\bar{\mathcal{D}}_\alpha$-generic over V. By elementarity, $k(G_{\bar{\mathcal{D}}_\alpha})$ is $k(\bar{\mathcal{D}}_\alpha)$-generic over $L[k(U)]$. Let $r_0 \in \bar{\mathcal{D}}_{\alpha,\beta}$ be defined by $r_0 = (\dot{q}_1 \restriction [\alpha, \beta))^{G_{\bar{\mathcal{D}}_\alpha}}$.

Work in $V[G_{\bar{\mathcal{D}}_\alpha}]$. Let $\langle N_\gamma \mid \gamma < \omega_1\rangle$ be a tower of countable elementary submodels of $H(\theta)^{V[G_{\bar{\mathcal{D}}_\alpha}]}$ for some sufficiently large regular cardinal θ with $r_0, \dot{D} \in N_0$. For every $\gamma < \omega_1$, let $\delta_\gamma = N_\gamma \cap \omega_1$. Let $E = \{\, \delta_\gamma \mid \gamma < \kappa \,\}$. Clearly, E is a club subset of κ. Let $\dot{E}$ be a $\bar{\mathcal{D}}_\alpha$-name for E.

Let $\left\langle \hat{N}_\gamma \mid \gamma < j(\kappa) \right\rangle = k(\langle N_\gamma \mid \gamma < \kappa\rangle)$. Note that for every $\gamma < \kappa$, $\hat{N}_\gamma = k(N_\gamma)$ and since N_γ is countable, $k(N_\gamma) = k``N_\gamma$. So, $\hat{N}_\gamma \cap \kappa = \delta_\gamma$. Recall $r_0 \in N_0$. So, $\operatorname{ht}(r_0) < \delta_0$.

Work in $L[k(U)][k(G_{\bar{\mathcal{D}}_\alpha})]$. Since $|\kappa|^{L[k(U)][k(G_{\bar{\mathcal{D}}_\alpha})]} = \aleph_0$, there exists an increasing cofinal sequence $\langle \kappa_n \mid n < \omega\rangle$ in κ. We shall define a decreasing sequence $\langle r_n \mid n < \omega\rangle$ in $k(\bar{\mathcal{D}}_{\alpha,\beta})$ and an increasing sequence $\langle \mu_n \mid n < \omega\rangle$ in κ so that $r_n \in \hat{N}_{\mu_n+1}$. r_0 is already given and $\mu_0 = \kappa_0$. Suppose that we have defined r_n so that $r_n \in \hat{N}_{\mu_n+1}$. In particular, $\operatorname{ht}(r_n) < \delta_{\mu_n+1}$. Let $\mu_{n+1} < \kappa$ be so large that $\mu_{n+1} > \max\{\, \kappa_{n+1}, \mu_n \,\}$ and $[\delta_{\mu_n+1}, \delta_{\mu_{n+1}}) \cap k(C)_\kappa \neq \varnothing$. Let $r'_n \leq r_n$ be defined by $r'_n(\xi) = r_n(\xi) \cup \{\, \delta_{\mu_{n+1}} + 1 \,\}$ for every $\xi \in N_{\mu_n+1} \cap \kappa^+$. Then, we have $r'_n \in \hat{N}_{\mu_{n+1}+1} \cap k(\bar{\mathcal{D}}_{\alpha,\beta})$. Hence, there exists an $r_{n+1} \leq r'_n$ such that $r_{n+1} \in \hat{N}_{\mu_{n+1}+1}$ and $r_{n+1} \Vdash k(\dot{D}) \cap [\delta_{\mu_n+1} + 1, \delta_{\mu_{n+1}+1}) \neq \varnothing$.

Define $r_\omega \in k(\bar{\mathcal{D}}_{\alpha,\beta})$ by letting $\mathrm{supp}(r_\omega) = \bigcup_{n<\omega} \mathrm{supp}(r_n)$ and for every $\xi \in \mathrm{supp}(r_\omega)$, $r_\omega(\xi) = \bigcup_{n<\omega} r_n(\xi) \cup \{\kappa\}$. It is easy to see that for every $\xi \in \mathrm{supp}(r_\omega)$, $k(C)_\kappa \not\subseteq^* r_\omega(\xi)$, so we have $r_\omega \in k(\bar{\mathcal{D}}_{\alpha,\beta})$. Notice that for every $\gamma < \kappa$ and $\xi \in N_\gamma \cap \kappa^+$, $\delta_\gamma \notin r_\omega(\xi)$. So, r_ω is disjoint from E which is a club subset of κ. Notice that $k(E) = E \cap \kappa$.

Let $\dot{r}_\omega$ a $k(\bar{\mathcal{D}}_\alpha)$-name for r_ω. Then, there exists $\left\langle \hat{p}, \dot{\hat{q}} \right\rangle \in k(G_{\bar{\mathcal{D}}_\alpha})$ such that $\left\langle \hat{p}, \dot{\hat{q}} \right\rangle$ forces that $\dot{r}_\omega(k(\alpha)) \cap k(\dot{E}) \cap \kappa = \varnothing$ and $\mathrm{ht}(\dot{r}_\omega) = \kappa$. Since $\langle p_1, \dot{q}_1 \rangle \in G_{\bar{\mathcal{D}}_\alpha}$, we have $k(\langle p_1, \dot{q}_1 \rangle) \in k(G_{\bar{\mathcal{D}}_\alpha})$. So, without loss of generality, we may assume $\left\langle \hat{p}, \dot{\hat{q}} \right\rangle \leq k(\langle p_1, \dot{q}_1 \rangle)$. Then, there exists a $\left\langle \hat{p}', \dot{\hat{q}}' \right\rangle \in k(\bar{\mathcal{D}}_\beta)$ such that $\left\langle \hat{p}', \dot{\hat{q}}' \restriction k(\alpha) \right\rangle \in k(G_{\bar{\mathcal{D}}_\alpha})$, $\left\langle \hat{p}', \dot{\hat{q}}' \restriction k(\alpha) \right\rangle \leq \left\langle \hat{p}, \dot{\hat{q}} \right\rangle$, and $\left\langle \hat{p}', \dot{\hat{q}}' \restriction k(\alpha) \right\rangle \Vdash \dot{\hat{q}} \restriction [k(\alpha), k(\beta)) \leq \dot{r}_\omega$. Then, $\left\langle \hat{p}', \dot{\hat{q}}' \right\rangle \Vdash k(\dot{D}_\alpha) \cap k(\dot{E}) \cap \kappa = \varnothing$. Recall $\dot{D}_\alpha$ is a $\bar{\mathcal{D}}_{\alpha+1}$-name for a club subset added at the α-stage of $\bar{\mathcal{D}}_{\alpha+1}$.

In V, we shall define $\mathcal{E}$ to be the set of all $\langle p, \dot{q} \rangle \in \bar{\mathcal{D}}_\beta$ such that for some $\gamma < \kappa$, $\langle p, \dot{q} \restriction \alpha \rangle \Vdash \gamma \in \dot{E} \cap \dot{q}(\alpha)$. It is easy to see that $\mathcal{E}$ is an open dense subset of $\bar{\mathcal{D}}_\beta$. Since $\mathcal{E} \in (H(\kappa^+))^V$, we have $\mathcal{E} \in L[k(U)]$.

Let $G_{k(\bar{\mathcal{D}}_\beta)}$ be $k(\bar{\mathcal{D}}_\beta)$-generic over $L[k(U)]$ extending $k(G_{\bar{\mathcal{D}}_\alpha})$ with $\left\langle \hat{p}', \dot{\hat{q}}' \right\rangle \in G_{k(\bar{\mathcal{D}}_\beta)}$ and work in $L[k(U)][G_{k(\bar{\mathcal{D}}_\beta)}]$. We shall show that $k^{-1}G_{k(\bar{\mathcal{D}}_\beta)}$ is not $\bar{\mathcal{D}}_\beta$-generic over $L[k(U)]$. Suppose that $k^{-1}G_{k(\bar{\mathcal{D}}_\beta)}$ is $\bar{\mathcal{D}}_\beta$-generic over $L[k(U)]$. Let $G_{\bar{\mathcal{D}}_\beta} = k^{-1}G_{k(\bar{\mathcal{D}}_\beta)}$.

Since $\mathcal{E}$ is an open dense subset of $\bar{\mathcal{D}}_\beta$, we have $\mathcal{E} \cap G_{\bar{\mathcal{D}}_\beta} \neq \varnothing$. Let $\langle p_2, \dot{q}_2 \rangle \in \mathcal{E} \cap G_{\bar{\mathcal{D}}_\beta}$. Then, there exists $\gamma < \kappa$ such that $\langle p_2, \dot{q}_2 \restriction \alpha \rangle \Vdash \gamma \in \dot{E} \cap \dot{q}_2(\alpha)$. Since $\langle p_2, \dot{q}_2 \rangle \in G_{\bar{\mathcal{D}}_\beta} = k^{-1}G_{\bar{\mathcal{D}}_\beta}$, we have $k(\langle p_2, \dot{q}_2 \rangle) \in G_{k(\bar{\mathcal{D}}_\beta)}$. Note $k(\langle p_2, \dot{q}_2 \restriction \alpha \rangle) \Vdash \gamma \in k(\dot{E}) \cap k(\dot{q}_2(\alpha)) \subseteq k(\dot{E}) \cap k(\dot{D}_\alpha)$. However, we know $\left\langle \hat{p}', \dot{\hat{q}}' \right\rangle \Vdash k(\dot{E}) \cap k(\dot{D}) \cap \kappa = \varnothing$. So, $k(\langle p_2, \dot{q}_2 \rangle)$ and $\left\langle \hat{p}', \dot{\hat{q}}' \right\rangle$ are incompatible though both belong to $G_{k(\bar{\mathcal{D}}_\beta)}$. This is a contradiction. □

Let $G * H$ be $P * \dot{Q}$-generic over V with $\langle p_0, \dot{q}_0 \rangle \in G * H$. Let $S = \dot{S}^{G*H}$ and $S' = (\dot{S}')^{G*H}$. Then, by assumption, $\mathrm{NS}_{\omega_1} \restriction S$ is precipitous. Thus, by Claim 5.2.1, $S' \in \mathrm{NS}_{\omega_1} \restriction S$. Since $S' \subseteq S$, S' is nonstationary. By Claim 5.2.2, S' is a stationary subset of S. This is a contradiction.

□

6. A tail club guessing sequence of order type $<\varepsilon$

We keep using P, Q, Q_α, R_α, $\left\langle C_\delta \mid \delta \in \bar{D} \right\rangle$, I, and I_α.

Lemma 6.1. *Assume $V = L[U]$. Let $G * H$ be $P * \dot{Q}$-generic over V. In $V[G][H]$, there exists no tail club guessing sequence $\vec{C}'$ of order type $<\varepsilon$ such that* $\mathrm{TCG}(\vec{C}')$ *is precipitous.*

Proof. In $V[G][H]$, suppose that $\vec{C}' = \left\langle C'_\delta \mid \delta \in \mathrm{dom}(\vec{C}') \right\rangle$ is a tail club guessing sequence of order type $\varepsilon' < \varepsilon$ on a stationary subset $\mathrm{dom}(\vec{C}')$ of $\kappa \cap \mathrm{Lim}$ and $\mathrm{TCG}(\vec{C}')$ is precipitous. Without loss of generality, by Proposition 2.4, we may assume that ε' is indecomposable. Let $\bar{D}'$ be the set of all limit ordinals $\gamma < \kappa$

such that there exists an unbounded subset C of γ that is τ-weakly tight and $\mathrm{otp}(C) = \varepsilon'$. Then, $\bar{D}'$ contains a club subset of ω_1. For each $\gamma \in \bar{D}' \setminus \mathrm{dom}(\vec{C}')$, by letting C'_δ be an arbitrary unbounded subset of γ that is τ-weakly tight and that has order type ε' without loss of generality, we may assume $\mathrm{dom}(\vec{C}') = \bar{D}'$. Let $\dot{\vec{C}}'$ be a $P * \dot{Q}$-name for $\vec{C}'$ and $\dot{C}'_\delta$ a $P * \dot{Q}$-name for C'_δ. Let $\langle p_0, \dot{q}_0\rangle \in G * H$ be so that $\langle p_0, \dot{q}_0\rangle$ forces all of these assumptions. Without loss of generality, we may assume there exists $\alpha < \kappa^+$ be such that $\dot{\vec{C}}'$ is a $\bar{\mathcal{D}}_\alpha$-name, $\langle p_0, \dot{q}_0\rangle \in \bar{\mathcal{D}}_\alpha$, and $\langle p_0, \dot{q}_0\rangle \Vdash \dot{X}_\alpha \in \dot{I}_\alpha$.

Let f be a function in V with domain κ such that for every $\gamma < \kappa$, $f(\gamma)$ is the set of all $s \in \bar{\mathcal{D}}_{\alpha+1}$ that forces if $(\nu_{\bar{\mathcal{D}}_{\alpha+1}}(\gamma))^{-1}(\dot{G}_{\bar{\mathcal{D}}_{\alpha+1}})$ is $\rho_{\bar{\mathcal{D}}_{\alpha+1}}(\gamma)$-generic over V, then $\dot{C}'_\gamma$ is not a fast club over $V[(\nu_{\bar{\mathcal{D}}_{\alpha+1}}(\gamma))^{-1}(\dot{G}_{\bar{\mathcal{D}}_{\alpha+1}})]$. Let $\dot{S}'$ be a $\bar{\mathcal{D}}_{\alpha+1}$-name for a subset of $\mathrm{dom}\,\vec{C}'$ such that for every $\gamma < \kappa$, $\langle p, \dot{q}\rangle \Vdash \gamma \in \dot{S}'$ if and only if $\langle p, \dot{q}\rangle \in f(\gamma)$.

Claim 6.1.1. $\langle p_0, \dot{q}_0\rangle \Vdash \dot{S}' \in \check{\mathrm{TCG}}(\vec{C}')$

Proof. Work in V. By way of contradiction, suppose otherwise. Then, there exists $\langle p_1, \dot{q}_1\rangle \le \langle p_0, \dot{q}_0\rangle$ such that $\langle p_1, \dot{q}_1\rangle \Vdash \dot{S}' \in \mathrm{TCG}(\dot{\vec{C}}')^+$.

Let $G * H$ be $P * \dot{Q}$-generic over V with $\langle p_1, \dot{q}_1\rangle \in G * H$. Let $S' = (\dot{S}')^{G*H}$. Let W be $\mathcal{P}(\kappa)/\mathrm{TCG}(\vec{C}')$-generic over $V[G][H]$ with $S' \in W$ and $k : V[G][H] \to M'$ the induced generic elementary embedding. By Lemma 3.2, we have $M' = L[k(U)][k(G)][k(H)]$. So, $k(G * H)$ is $k(P * \dot{Q})$-generic over $L[k(U)]$. Since $S' \in W$, we have $\kappa \in k(S') = k(\dot{S}')^{k(G*H)}$.

Since W is $\mathcal{P}(\kappa)/\mathrm{TCG}(\vec{C}')$-generic over $V[G][H]$, for every club subset E of κ lying in $V[G][H]$, $\{\, \delta \in \mathrm{dom}(\vec{C}') \mid C'_\delta \subseteq^* E \,\} \in W$. So, $k(C')_\kappa \subseteq^* k(E)$. Since $k(E) \cap \kappa = E$ and $k(C')_\kappa$ is an unbounded subset of κ, we have $k(C')_\kappa \subseteq^* E$. Thus, $k(C')_\kappa$ is a fast club over $V[G][H]$.

Let $G_{\bar{\mathcal{D}}_{\alpha+1}} = (G * H) \cap \bar{\mathcal{D}}_{\alpha+1}$. Then, we have $L[k(U)][k(G)][k(H \cap Q_{\alpha+1})] = L[k(U)][k(G_{\bar{\mathcal{D}}_{\alpha+1}})]$. Let $G_{k(\bar{\mathcal{D}}_{\alpha+1})} = k(G_{\bar{\mathcal{D}}_{\alpha+1}})$. We have $k^{-1}G_{k(\bar{\mathcal{D}}_{\alpha+1})} = G_{\bar{\mathcal{D}}_{\alpha+1}}$, which is $\bar{\mathcal{D}}_{\alpha+1}$-generic over V. Since $L[k(U)] \subseteq V$, $k^{-1}G_{k(\bar{\mathcal{D}}_{\alpha+1})}$ is $\bar{\mathcal{D}}_{\alpha+1}$-generic over $L[k(U)]$.

Since $\kappa \in k(S') = k(\dot{S}')^{k(G*H)}$, there exists $\hat{s} \in k(G*H)$ such that $\hat{s} \Vdash \kappa \in k(\dot{S}')$. Since $\dot{S}'$ is $\bar{\mathcal{D}}_{\alpha+1}$-name, $k(\dot{S}')$ is a $k(\bar{\mathcal{D}}_{\alpha+1})$-name. So, we may assume $\hat{s} \in k(\bar{\mathcal{D}}_{\alpha+1})$. Since $\hat{s} \in k(\bar{\mathcal{D}}_{\alpha+1})$, there exists a function $g : \kappa \to \bar{\mathcal{D}}_{\alpha+1}$ lying in $V[G][H]$ such that

$[g]_W = \hat{s}$. Then,

$$\begin{aligned}
&\hat{s} \Vdash \kappa \in k(\dot{S}') \\
&\iff [g]_W \Vdash \kappa \in k(\dot{S}') \\
&\iff \{\, \gamma < \kappa \mid g(\gamma) \Vdash \gamma \in \dot{S}' \,\} \in W \\
&\iff \{\, \gamma < \kappa \mid g(\gamma) \in f(\gamma) \,\} \in W \\
&\iff \big\{\gamma < \kappa \mid g(\gamma) \text{ forces if } (\nu_{\bar{\mathcal{D}}_{\alpha+1}}(\gamma))^{-1}(\dot{G}_{\bar{\mathcal{D}}_{\alpha+1}}) \text{ is } \rho_{\bar{\mathcal{D}}_{\alpha+1}}(\gamma)\text{-generic over } V, \\
&\qquad\qquad \text{then } \dot{C}'_\gamma \text{ is not a fast club over } V[(\nu_{\bar{\mathcal{D}}_{\alpha+1}}(\gamma))^{-1}(\dot{G}_{\bar{\mathcal{D}}_{\alpha+1}})]\big\} \in W \\
&\iff [g]_W \text{ forces if } k^{-1}\dot{G}_{k(\bar{\mathcal{D}}_{\alpha+1})} \text{ is } \bar{\mathcal{D}}_{\alpha+1}\text{-generic over } L[k(U)], \\
&\qquad \text{then } k(\dot{C}')_\delta \text{ is not a fast club over } L[k(U)][k^{-1}\dot{G}_{k(\bar{\mathcal{D}}_{\alpha+1})}] \\
&\iff \hat{s} \text{ forces if } k^{-1}\dot{G}_{k(\bar{\mathcal{D}}_{\alpha+1})} \text{ is } \bar{\mathcal{D}}_{\alpha+1}\text{-generic over } L[k(U)], \\
&\qquad \text{then } k(\dot{C}')_\delta \text{ is not a fast club over } L[k(U)][k^{-1}\dot{G}_{k(\bar{\mathcal{D}}_{\alpha+1})}].
\end{aligned}$$

So, in $L[k(U)][k(G)][k(H)]$, if $k^{-1}G_{k(\bar{\mathcal{D}}_{\alpha+1})}$ is $\bar{\mathcal{D}}_{\alpha+1}$-generic over $L[k(U)]$, then $k(C')_\delta$ is not a fast club over $L[k(U)][k^{-1}G_{k(\bar{\mathcal{D}}_{\alpha+1})}]$. However, we have already shown that $k^{-1}G_{k(\bar{\mathcal{D}}_{\alpha+1})}$ is $\bar{\mathcal{D}}_{\alpha+1}$-generic over $L[k(U)]$ and $k(C')_\delta$ is a fast club over $L[k(U)][k^{-1}G_{k(\bar{\mathcal{D}}_{\alpha+1})}]$. This is a contradiction. $\square$

Claim 6.1.2. $\langle p_0, \dot{q}_0\rangle \Vdash_{P*\dot{Q}} \dot{S}' \in \mathrm{TCG}(\vec{C}')^+$.

Proof. Suppose not. Then, there exist $\langle p_1, \dot{q}_1\rangle \in P*\dot{Q}$ and a $P*\dot{Q}$-name $\dot{D}$ for a club subset of κ such that $\langle p_1, \dot{q}_1\rangle \leq \langle p_0, \dot{q}_0\rangle$ and $\langle p_1, \dot{q}_1\rangle \Vdash \forall\gamma \in \dot{S}'(C'_\gamma \nsubseteq^* \dot{D})$. Without loss of generality, we may assume that there exists $\beta < \kappa^+$ such that $\beta > \alpha$, $\langle p_1, \dot{q}_1\rangle \in \bar{\mathcal{D}}_\beta$ and $\dot{S}'$ is $\bar{\mathcal{D}}_\beta$-name.

Let $G*H$ be $P*\dot{Q}$-generic over V with $\langle p_1, \dot{q}_1\rangle \in G*H$. Work in $V[G][H]$. By assumption, $\mathrm{TCG}(\vec{C}')$ is precipitous. Let W be $\mathcal{P}(\kappa)/\mathrm{TCG}(\vec{C}')$-generic over $V[G][H]$ and $k : V[G][H] \to L[k(U)][k(G)][k(H)]$ the induced generic elementary embedding. For every club subset E of κ lying in $V[G][H]$, by definition, we have $\{\, \delta \in \mathrm{dom}(\vec{C}') \mid C'_\delta \subseteq E \,\} \in W$. Thus, $k(C')_\kappa \subseteq^* k(E)$. Since $k(E)\cap\kappa = E$, we have $k(C')_\kappa \subseteq^* E$, i.e. $k(C')_\kappa$ is a fast club over $V[G][H]$.

Let $G_{\bar{\mathcal{D}}_\alpha} = (G*H)\cap\bar{\mathcal{D}}_\alpha$. Then, $G_{\bar{\mathcal{D}}_\alpha}$ is $\bar{\mathcal{D}}_\alpha$-generic over V. By elementarity, $k(G_{\bar{\mathcal{D}}_\alpha})$ is $k(\bar{\mathcal{D}}_\alpha)$-generic over $L[k(U)]$. Define $r_0 \in \bar{\mathcal{D}}_{\alpha,\beta}$ by letting $r_0 = (\dot{q}_1 \restriction [\alpha,\beta))^{G_{\bar{\mathcal{D}}_\alpha}}$.

Work in $V[G_{\bar{\mathcal{D}}_\alpha}]$. Let $\langle N_\gamma \mid \gamma < \kappa\rangle$ be a tower of countable elementary submodels of $H(\theta)^{V[G_{\bar{\mathcal{D}}_\alpha}]}$ for some sufficiently large regular cardinal θ with $r_0, \vec{C}', \dot{D}, \tau \in N_0$. For every $\gamma < \kappa$, let $\delta_\gamma = N_\gamma \cap \kappa$ and $E = \{\, \delta_\gamma \mid \gamma < \kappa \,\}$.

Work in $L[k(U)][k(G)][k(H)]$. Let $\left\langle \hat{N}_\gamma \mid \gamma < k(\kappa)\right\rangle = k(\langle N_\gamma \mid \gamma < \kappa\rangle)$. Then, $\left\langle \hat{N}_\gamma \mid \gamma < k(\kappa)\right\rangle$ is a tower of countable elementary submodels of $H(k(\theta))^{L[k(U)][k(\bar{\mathcal{D}}_\alpha]]}$. As in the proof of Lemma 5.2, for every $\gamma < \kappa$, $\hat{N}_\gamma = k(N_\gamma) = k``N_\gamma$, $\hat{N}_\gamma \cap \kappa = \delta_\gamma$. Since $r_0 \in N_0$, we have $\mathrm{ht}(r_0) < \delta_0$.

Since $k(C')_\kappa$ is a fast club over $V[G][H]$, we have $k(C')_\kappa \subseteq^* E$. Let $\zeta < \kappa$ be so that $k(C')_\kappa \setminus \zeta \subseteq E$. Let F be the set of all $\gamma < \kappa$ such that $\delta_\gamma \in k(C')_\kappa \setminus \zeta$.

Subclaim 6.1.2.1. For every $\gamma \in \mathrm{nacc}(F)$, $F \cap \gamma \in \hat{N}_\gamma$.

Proof. Let $\gamma \in \mathrm{nacc}(F)$. If $\gamma = \min(F)$, then we have $F \cap \gamma = \varnothing \in \hat{N}_\gamma$. Suppose $\gamma > \min(F)$. Since $\gamma \in \mathrm{nacc}(F)$, there exists $\gamma' \in F$ such that $\gamma' = \max(F \cap \gamma)$. Since $\vec{C}'$ is τ-weakly tight, we have $k(C')_\kappa \cap \delta_\gamma \in \hat{N}_\gamma$. So, $(k(C')_\kappa \cap \delta_\gamma) \setminus \zeta \in \hat{N}_\gamma$. Notice $(k(C')_\kappa \cap \delta_\gamma) \setminus \zeta = (k(C')_\kappa \setminus \zeta) \cap (\delta_{\gamma'} + 1)$. Since $\left\langle \hat{N}_\xi \mid \xi < k(\kappa) \right\rangle$ is a tower, $\langle \delta_\xi \mid \xi \leq \gamma' \rangle \in \hat{N}_\gamma$. So, $F \cap \gamma = \{ \xi \leq \gamma' \mid \delta_\xi \in k(C')_\kappa \setminus \zeta \}$ can be computed correctly in $\hat{N}_\gamma$. □

Note that $k(C)_{\delta_\gamma} = k(C)_{k(\delta_\gamma)} = k(C_{\delta_\gamma}) = C_{\delta_\gamma}$.

Subclaim 6.1.2.2. For every limit ordinal $\gamma < \kappa$, $C_{\delta_\gamma} \not\subseteq^* \{ \delta_\xi \mid \xi \in F \cap \gamma \}$.

Proof. Note that since $\bar{D}$ is a club subset of κ and $\bar{D} \in \hat{N}_0$, for every $\gamma < \kappa$, we have $\delta_\gamma = \hat{N}_\gamma \cap \kappa \in \bar{D}$. Thus, C_{δ_γ} exists. Since $\vec{C}'$ has order type ε', we have $\mathrm{otp}(\{ \delta_\xi \mid \xi \in F \cap \gamma \}) = \mathrm{otp}((k(C')_\kappa \setminus \zeta) \cap \delta_\gamma) \leq \varepsilon' < \varepsilon$. Since $\vec{C}$ has order type ε, we have $\mathrm{otp}(C_{\delta_\gamma}) = \varepsilon$. Since ε is indecomposable, we have $C_{\delta_\gamma} \not\subseteq^* \{ \delta_\xi \mid \xi \in E \cap \gamma \}$. □

Notice that $Q_{\alpha,\beta}$ is a countable support iteration of forcing notions that are trivial or of the form $R_\alpha = P(\vec{C}, \kappa \setminus X_\alpha)$. So, $k(Q_{\alpha,\beta})$ is a countable support iteration of forcing notions that are trivial or of the form $k(R_\alpha) = P(k(\vec{C}), k(\kappa \setminus X_\alpha))$. $k(\bar{\mathcal{D}}_{\alpha,\beta})$ is a dense subset of $k(Q_{\alpha,\beta})$. By Lemma 2.19, there exists $r' \leq k(r_0)$ such that for every $\gamma < \kappa$, if $\gamma \in F$, then r' is totally $(\hat{N}_\gamma, k(\bar{\mathcal{D}}_{\alpha,\beta}))$-generic; and if $\gamma \notin F$, then for every $\xi \in \hat{N}_\gamma \cap [k(\alpha), k(\beta)))$, $\delta_\gamma \notin r'(\xi)$. Without loss of generality, we may assume that $\mathrm{ht}(r') = \kappa$.

For every $\gamma \in F$, since r' is totally $(\hat{N}_\gamma, k(R_\alpha))$-generic and $\dot{D} \in N_0 \subseteq N_\gamma$, we have $r' \Vdash \delta_\gamma \in k(\dot{D})$. Thus, $r' \Vdash k(C')_\kappa \setminus \zeta \subseteq k(\dot{D})$. Also, since $\gamma \notin F$ implies $\delta_\gamma \notin r'(k(\alpha))$, we have $E \cap r'(k(\alpha)) = \{ \delta_\gamma \mid \gamma < \kappa \} \cap r'(k(\alpha)) \subseteq \{ \delta_\gamma \mid \gamma \in F \}$. So, $\mathrm{otp}(E \cap r'(k(\alpha))) \leq \mathrm{otp}(F) = \varepsilon'$. Note $k(E) \cap \kappa = E$. Since $\max(r'(k(\alpha))) = \mathrm{ht}(r') = \kappa$, we have $r' \Vdash \mathrm{otp}(k(E) \cap k(\dot{D}_\alpha) \cap \kappa) \leq \varepsilon'$.

Let $\dot{r}'$ be a $k(\bar{\mathcal{D}}_\alpha)$-name for r'. Then, there exists $\left\langle \hat{p}, \dot{\hat{q}} \right\rangle \in k(\bar{\mathcal{D}}_\alpha)$ such that $\mathrm{ht}\left(\left\langle \hat{p}, \dot{\hat{q}} \right\rangle\right) = \kappa$, $\left\langle \hat{p}, \dot{\hat{q}} \right\rangle$ forces that $\dot{r}' \leq k(\dot{q}_1 \restriction [\alpha, \beta))$cd and $\dot{r}' \Vdash k(\dot{C}')_\kappa \setminus \zeta \subseteq k(\dot{D}_\alpha) \wedge \mathrm{otp}(k(E) \cap k(\dot{D}_\alpha) \cap \kappa) \leq \varepsilon'$. Here $\dot{D}_\alpha$ is identified with a $\bar{\mathcal{D}}_\alpha$-name for a $\dot{R}_\alpha$-name. Since $\langle p_1, \dot{q}_1 \rangle \in G * H$, we have $\langle p_1, \dot{q}_1 \restriction \alpha \rangle \in G_{\bar{\mathcal{D}}_\alpha}$ and hence $k(\langle p_1, \dot{q}_1 \restriction \alpha \rangle) \in k(G_{\bar{\mathcal{D}}_\alpha})$. So, without loss of generality, we may assume $\left\langle \hat{p}, \dot{\hat{q}} \right\rangle \leq k(\langle p_1, \dot{q}_1 \restriction \alpha \rangle)$. There exists a $\left\langle \hat{p}', \dot{\hat{q}}' \right\rangle \in k(\bar{\mathcal{D}}_\beta)$ such that $\left\langle \hat{p}', \dot{\hat{q}}' \restriction k(\alpha) \right\rangle \in k(G_{\bar{\mathcal{D}}_\alpha})$, $\left\langle \hat{p}', \dot{\hat{q}}' \restriction k(\alpha) \right\rangle \leq \left\langle \hat{p}, \dot{\hat{q}} \right\rangle$, and $\left\langle \hat{p}', \dot{\hat{q}}' \restriction k(\alpha) \right\rangle \Vdash \dot{\hat{q}} \restriction [k(\alpha), k(\beta)) \leq \dot{r}'$. Then, $\left\langle \hat{p}', \dot{\hat{q}}' \right\rangle \Vdash k(\dot{C}'_\kappa) \setminus \zeta \subseteq k(\dot{D}) \wedge \mathrm{otp}(k(E) \cap k(\dot{D}_\alpha) \cap \kappa) \leq \varepsilon'$. Also, since $\left\langle \hat{p}', \dot{\hat{q}}' \restriction k(\alpha) \right\rangle \leq \left\langle \hat{p}, \dot{\hat{q}} \right\rangle \leq k(\langle p_1, \dot{q}_1 \restriction \alpha \rangle)$ and $\left\langle \hat{p}', \dot{\hat{q}}' \restriction k(\alpha) \right\rangle \Vdash \dot{\hat{q}} \restriction [k(\alpha), k(\beta)) \leq \dot{r}' \leq k(\dot{q}_1 \restriction [\alpha, \beta))$, we have $\left\langle \hat{p}', \dot{\hat{q}}' \right\rangle \leq k(\langle p_1, \dot{q}_1 \rangle)$.

In V, we shall define $\mathcal{E}$ to be the set of all $\langle p, \dot{q} \rangle \in \bar{\mathcal{D}}_\beta$ such that for some $\gamma < \kappa$, $\langle p, \dot{q} \rangle \Vdash \mathrm{otp}(\dot{E} \cap \dot{D}_\alpha \cap \gamma) > \varepsilon'$. It is easy to see that $\mathcal{E}$ is an open dense subset of $\bar{\mathcal{D}}_\beta$.

Let $G_{k(\bar{\mathcal{D}}_\beta)}$ be $k(\bar{\mathcal{D}}_\beta)$-generic over $L[k(U)]$ extending $k(G_{\bar{\mathcal{D}}_\alpha})$ with $\left\langle \hat{p}', \dot{\hat{q}}' \right\rangle \in G_{k(\bar{\mathcal{D}}_\beta)}$. We shall show that $k^{-1}G_{k(\bar{\mathcal{D}}_\beta)}$ is not $\bar{\mathcal{D}}_\beta$-generic over $L[k(U)]$. Suppose otherwise, i.e. $k^{-1}G_{k(\bar{\mathcal{D}}_\beta)}$ is $\bar{\mathcal{D}}_\beta$-generic over $L[k(U)]$.

Since $\mathcal{E}$ is an open dense subset of $\bar{\mathcal{D}}_\beta$ in V, we have $\mathcal{E} \cap k^{-1}G_{k(\bar{\mathcal{D}}_\beta)} \neq \varnothing$. Let $\langle p_2, \dot{q}_2 \rangle \in \mathcal{E} \cap k^{-1}G_{k(\bar{\mathcal{D}}_\beta)}$. Since $\langle p_2, \dot{q}_2 \rangle \in k^{-1}G_{k(\bar{\mathcal{D}}_\beta)}$, we have $k(\langle p_2, \dot{q}_2 \rangle) \in G_{k(\bar{\mathcal{D}}_\beta)}$. Since $\langle p_2, \dot{q}_2 \rangle \in \mathcal{E}$, there exists $\gamma < \kappa$ such that $\langle p_2, \dot{q}_2 \rangle \Vdash \operatorname{otp}(\dot{E} \cap \dot{D}_\alpha \cap \gamma) > \varepsilon'$. So, $k(\langle p_2, \dot{q}_2 \rangle) \Vdash \operatorname{otp}(k(\dot{E}) \cap k(\dot{D}_\alpha) \cap \gamma) > \varepsilon'$. However, $\left\langle \hat{p}', \dot{\hat{q}}' \right\rangle \in G_{k(\bar{\mathcal{D}}_\beta)}$ and $\left\langle \hat{p}', \dot{\hat{q}}' \right\rangle \Vdash \operatorname{otp}(k(E) \cap k(\dot{D}_\alpha) \cap \kappa) \leq \varepsilon'$. So, $k(\langle p_2, \dot{q}_2 \rangle)$ and $\left\langle \hat{p}', \dot{\hat{q}}' \right\rangle$ are incompatible though both belong to $G_{k(\bar{\mathcal{D}}_\beta)}$. This is a contradiction. □

The previous two claims contradict each other. □

This finishes the proof of Theorem 4.1.

7. Open problems

While Theorem 4.1 answers some questions asked in [4], it leaves the following question open.

Question 1. Is it consistent that NS_{ω_1} is precipitous but there is a tail club guessing sequence $\vec{C}$ on ω_1 such that $\mathrm{TCG}(\vec{C})$ is not precipitous?

In [6], Jech, Magidor, Mitchell, and Prikry constructed from a measurable cardinal a model in which NS_{ω_1} is precipitous. We can show that there is a tail club guessing sequence in this model. If the tail club guessing ideal associated with it is not precipitous, then the previous question is solved negatively. Otherwise, it solves the following question, which is also interesting.

Question 2. What is the consistency strength that NS_{ω_1} is precipitous and there is a precipitous tail club guessing ideal on ω_1?

Note that the author constructed from a Woodin cardinal a model in which NS_{ω_1} and all tail club guessing ideals on ω_1 are precipitous. Thus, the existence of one Woodin cardinal is an upper bound of the consistency strength. However, it is expected to be much lower.

The following question is not solved in this paper either.

Question 3. Is it consistent that there are two tail club guessing sequences $\vec{C}$ and $\vec{C}'$ such that $\mathrm{TCG}(\vec{C})$ is precipitous, $\mathrm{TCG}(\vec{C}')$ is not precipitous, and $\vec{C}$ has smaller order type than $\vec{C}'$?

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Email address: `ishiut@miamioh.edu`

Department of Mathematics,, Miami University,, Oxford, OH 45056, USA