EVERY CANCELLATIVE TOPOLOGICAL SEMIGROUP ON THE FINITE PRODUCT OF CONNECTED LINEARLY ORDERED TOPOLOGICAL SPACES IS SEPARABLE

TETSUYA ISHIU

Abstract. We shall prove that every cancellative topological semigroup on the finite product of connected linearly ordered topological spaces is separable.

1. Introduction

When K is a linearly ordered set, K is often given the order topology. The topological space obtained in this way is called a *linearly ordered topological space*, which is abbreviated as LOTS. The most typical example of connected LOTS is the real line. So, the finite product of connected LOTS can be considered as a generalization of Euclidean spaces. The similarities and differences between Euclidean spaces and the finite product of connected LOTS have been studied in [9], [5], [6], and [7].

In particular, the author proved the following theorem in [7].

Theorem 1.1 (Ishiu, [7]). *Let n be a positive integer, and K_i a connected LOTS. If there is a cancellative topological semigroup on $\prod_{i<n} K_i$, then for every $i < n$ and $p \in K_i$, if p is not the maximum or minimum element of K_i, then p has a separable open neighborhood in K_i*

The goal of this paper is to improve this theorem to show that for every $i < n$, K_i is separable. Notice that the only separable connected LOTS are $\{ 0 \}$, $\mathbb{R}$, $[0, 1]$, $[0, 1)$, and $(0, 1]$ up to isomorphism.

Some special cases are already known. The following theorem was proved by J. Aczel in [1] and the proof was simplified by R. Craigen and Z. Páles in [3].

Theorem 1.2 (Aczel [1]). *Every cancellative topological semigroup on a connected LOTS is order- and semigroup-isomorphic to a subsemigroup of $(\mathbb{R}, +)$.*

So, in case of dimension 1, even the structure of the topological semigroup is fairly restricted.

In case of higher dimensions, we can observe the following facts.

- Every topological group on the finite product of connected LOTS is locally Euclidean.

Date: January 3, 2026.

- Every cancellative topological monoid on the finite product of connected LOTS without end points is a locally Euclidean group.

In fact, in both cases, they are Lie groups. See Section 3 for proofs and more information. So, the essential improvements of this article is that even when the semigroup may not be a monoid or some K_i may have end points, we can show that K_i is separable, not only locally.

2. Preparation

In this section, we shall remind some notations and introduce several theorems that are used to prove the main theorem. Most of our notations are standard.

For set-theoretic notations, we refer to standard textbooks such as K. Kunen [8]. We adopt von Neumann ordinals. So, every ordinal is represented by the set of all ordinals that are less than it. For example, if α is an ordinal and A is a set of ordinals, $A \cap \alpha$ is the set of all ordinals in A that are less than α. We can prove that for every ordinal α, the least ordinal greater than α is $\alpha \cup \{ \alpha \}$, which is denoted by $S(\alpha)$. We say that an ordinal α is a *successor ordinal* if and only if there exists an ordinal β such that $\alpha = S(\beta)$. Otherwise, we say that α is a *limit ordinal* if and only if it is not a successor ordinal. Recall that ω_1 is the least uncountable ordinal.

We shall use some ordinal arithmetic. For details, see [8]. We can define addition and multiplication on ordinal numbers. When applied to finite ordinals, they coincide with the ordinary addition and multiplication of natural numbers. However, they are associative, but not commutative. We shall describe the properties that are used in this article. Every ordinal α can be uniquely expressed as the sum of a limit ordinal and a finite ordinal, i.e., for every ordinal α, there exists a unique pair of a limit ordinal δ and a finite ordinal n such that $\alpha = \delta + n$. When δ is a limit ordinal, 2δ is the order type of $\omega \times 2$ in the lexicographical ordering, so equal to δ. It should not be confused with $\delta \cdot 2$, which is equal to $\delta + \delta$ that is strictly bigger than δ unless $\delta = 0$. Suppose that an ordinal β is decomposed as $\beta = \delta + n$ where δ is a limit ordinal and n is a finite ordinal. Then, $2\beta = \delta + 2n$. Thus, every ordinal α can be expressed as 2β or $2\beta + 1$ for some ordinal β. In addition, the successor of $2\beta + 1$ is $2\beta + 2 = 2(\beta + 1)$. Notice its similarity to the behavior of even and odd natural number. Also, it is easy to see that $2\beta \geq \beta$. Moreover, for every ordinal β and every limit ordinal γ, if $\beta < \gamma$, then we have $2\beta < \gamma$.

Let A be a set of ordinal and $D \subseteq A$. We say that D is *unbounded in A* if and only if for every $\alpha \in A$, there exists $\beta \in D$ such that $\alpha \leq \beta$. We say that D is *closed in A* if and only if for every limit ordinal $\alpha \in A$, if $D \cap \alpha$ is unbounded in α, then $\alpha \in D$. Suppose that γ is an ordinal such that $A \subseteq \gamma$. Give the order topology to γ, and A is considered as a subspace of γ. Then, D is closed in A if and only if D is closed in the topological space A. We say that D is a *club subset of A* if and only if D is unbounded and closed in

A. Notice that if D is a club subset of A and E is a club subset of D, then E is a club subset of A.

Let X be a topological space, and D a club subset of a limit ordinal η. We say that a sequence $\langle x_\alpha \mid \alpha \in D\rangle$ is *continuous* if and only if the function $\alpha \mapsto X$ is continuous, when the topology of D is given as in the previous paragraph. In other words, $\langle x_\alpha \mid \alpha \in D\rangle$ is continuous if and only if for every limit ordinal $\alpha \in D$, if $D \cap \alpha$ is unbounded in α, then the sequence $\langle x_\beta \mid \beta \in D \cap \alpha\rangle$ converges to x_α.

Let $f : A \to B$ be a function. When $S \subseteq A$, we shall write $f^{\to}S$ to mean the image of S under f. When $T \subseteq B$, we shall write $f^{\leftarrow}T$ to mean the preimage of T under f.

When S is a semigroup and $a \in S$, we define the functions $l_a : S \to S$ and $r_a : S \to S$ by $l_a(x) = ax$ and $r_a(x) = xa$. If S is a topological semigroup, by definition, both l_a and r_a are continuous. If S is in addition cancellative, both l_a and r_a are injective.

Recall the following theorem that was proved in [7].

Theorem 2.1 (Ishiu [7])**.** *Let n be a positive integer, K_i a connected LOTS with no end points and $|K_i| \geq 2$ for each $i < n$, and L_j a connected LOTS for each $j < n$. Then, every continuous injective function from an open subset of $\prod_{i<n} K_i$ into $\prod_{j<n} L_j$ is an open map.*

Thus, every cancellative topological semigroup S on the finite product of connected LOTS with no end points has open shifts, i.e. for every $a \in S$, both l_a and r_a are open maps.

Even when some K_i has end points, we can obtain the following partial results.

Corollary 2.2. *Let n be a positive integer, K_i a connected and $|K_i| \geq 2$ for each $i < n$, and L_j a connected LOTS for each $j < n$. Let f be a continuous injection from a nonempty open set of $\prod_{i<n} K_i$ into $\prod_{j<n} L_j$. Suppose that for every $x \in \operatorname{dom}(f)$ and $i < n$, $x(i)$ is neither the maximum or minimum element of K_i. Then, f is an open map.*

For LOTS, J. R. Munkres [11] is a good reference for basic information. The following facts are used in this article.

Fact 2.3. Let K be a connected LOTS.

- K is self-dense and has the least upper bound property.
- Every bounded closed interval in K is compact.

We shall use the interval notations for a LOTS K. For example, for all $p, q \in K$, (p, q) is the set of all $r \in K$ such that $p < r < q$. Notice that for all connected LOTS K and L, K and L are homeomorphic if and only if K is isomorphic to L or the reverse of L. Thus, K is homeomorphic to $\mathbb{R}$ if and only if K is isomorphic to $\mathbb{R}$ as linearly ordered sets.

We shall use the notion of coinitial and cofinal subsets of linearly ordered sets. Let K be a linearly ordered set. We say that a decreasing sequence

$\langle a_\alpha \mid \alpha < \eta \rangle$ in K (where η is some ordinal) is a *coinitial sequence in* K if and only if for every $p \in K$, there exists $\alpha < \eta$ such that $a_\alpha \leq p$. Similarly, We say that an increasing sequence $\langle a_\alpha \mid \alpha < \eta \rangle$ in K (where η is some ordinal) is a *cofinal sequence in* K if and only if for every $p \in K$, there exists $\alpha < \eta$ such that $p \leq a_\alpha$. It is well-known that for every linearly ordered set K, there exist a coinitial sequence $\langle a_\alpha \mid \alpha < \eta \rangle$ such that η is a cardinal in von Neumann's sense, i.e., η is an ordinal that does not have the same cardinality with any ordinal below it. Similarly, there exists a cofinal sequence $\langle b_\alpha \mid \alpha < \zeta \rangle$ such that ζ is a cardinal. In particular, if K has no uncountable strictly monotone sequence nor end points, then there exist a cofinal sequence and a coinitial sequence of length ω.

3. Backgrounds

In this section, we shall give several known results and examples.

For each positive integer n, $\mathbb{R}^n$ with ordinary vector addition (denoted by $(\mathbb{R}^n, +)$) is a topological group on the finite product of connected LOTS. This is not the unique topological group on a Euclidean space in each dimension. For example, the group $\mathrm{Aff}_+(1)$ of orientation-preserving affine transformation of the real line. Recall that $\mathrm{Aff}_+(1)$ is the group of transformations $x \mapsto ax + b$ with $a > 0$. It is technically defined on $\mathbb{R}^+ \times \mathbb{R}$, but it is clearly homeomorphic to $\mathbb{R}^2$. In addition, it is not isomorphic to $(\mathbb{R}^2, +)$ as $\mathrm{Aff}_+(1)$ is not commutative.

The following result was proved in [7].

Theorem 3.1 (Ishiu [7])**.** *Every cancellative topological monoid on the finite product of connected LOTS without end points is a locally separable group.*

For topological groups, recall the following theorem, proved by A. Gleason [4] and D. Montgomery and L. Zippen [10] and considered as the solution of Hilbert's fifth problem.

Theorem 3.2 (Gleason, Montgomery, and Zippen)**.** *Every locally Euclidean group is a Lie group.*

We can use this theorem to show the following.

Fact 3.3. Every topological group on the finite product of connected LOTS is a Lie group.

Proof. Let n be a positive integer and K_i a connected LOTS for each $i < n$. Suppose that there is a topological group on $\prod_{i<n} K_i$. Then, $\prod_{i<n} K_i$ is homogeneous and hence for every $i < n$, K_i has no end points. By Theorem 3.1, $\prod_{i<n} K_i$ is locally separable. It is easy to see that it implies $\prod_{i<n} K_i$ is locally Euclidean. By Theorem 3.2, $\prod_{i<n} K_i$ is a Lie group. □

Thus, the class of all topological group on the finite product of connected LOTS coincides with the class of all Lie groups on a Euclidean space. So, we can apply the results on Lie groups. For example, we can show that

$(\mathbb{R}^2, +)$ and $\mathrm{Aff}_+(1)$ are only two topological groups on the product of two connected LOTS up to isomorphism, because they are only Lie groups on $\mathbb{R}^2$.

Similarly, we can observe the following.

Fact 3.4. Every cancellative topological monoid on the finite product of connected LOTS without end points is a Lie group.

Proof. Let n be a positive integer and K_i a connected LOTS without end points for each $i < n$. Suppose that there is a cancellative topological monoid on $\prod_{i<n} K_i$. Since we assumed that each connected LOTS has no end points, by Theorem 3.1, $\prod_{i<n} K_i$ is a group. Thus, by Fact 3.3, $\prod_{i<n} K_i$ is a Lie group. $\square$

Notice that the assumption that each connected LOTS has no end points cannot be removed, since $[0, \infty)$ with ordinary addition is a cancellative topological monoid that is not a group. Moreover, the assumption that the semigroup is a monoid cannot be removed either because $(0, \infty)$ with ordinary addition is a cancellative topological semigroup on a connected LOTS without end points.

4. Proof

In this section, we shall prove the main theorem and discuss its consequences.

We need the following lemma.

Lemma 4.1. *Let K be a connected LOTS with no uncountable strictly monotone sequence such that for every $p \in K$, if p is neither the maximum or minimum element of K, p has a separable open neighborhood. Then, K is separable.*

Proof. First, we shall assume that K has neither the maximum or minimum element of K. By the assumption, K is locally separable. Since there is no uncountable strictly monotone sequence, there exists a strictly decreasing sequence $\langle a_n \mid n < \omega \rangle$ that is coinitial in K. Also, there exists a strictly increasing sequence $\langle b_n \mid n < \omega \rangle$ that is cofinal in K. Note that $K = \bigcup_{n<\omega} [a_n, b_n]$. Recall that every closed bounded interval in a connected LOTS is compact. For every $n < \omega$, since K is locally separable and $[a_n, b_n]$ is compact, $[a_n, b_n]$ is separable. Since K is the countable union of separable subspaces, K is separable.

Now, consider the case when K may have the maximum or minimum element. Let K' be the LOTS obtained by removing the maximum and minimum elements of K if they exist. By the result above, K' is separable. In addition, K is the closure of K'. So, K' is separable. $\square$

Now, we are ready to prove the following main lemma.

Theorem 4.2. *Every cancellative topological semigroup on the finite product of connected LOTS is separable.*

Proof. Let n be a positive integer and for each $i < n$, let K_i be a connected LOTS. Let $\vec{K} = \prod_{i<n} K_i$. Suppose that there is a cancellative topological semigroup operation $\cdot$ on $\vec{K}$. Suppose that $\vec{K}$ is not separable. Then, there exists $i < n$ such that K_i is not separable. Without loss of generality, we may assume that K_0 is not separable. By Lemma 4.1, K_0 has an uncountable strictly monotone sequence. By considering the reverse if it is necessary, without loss of generality, we may assume it has an uncountable strictly increasing sequence. Let $\langle p_\alpha \mid \alpha < \omega_1 \rangle$ be a strictly increasing continuous sequence on K_0. Here, the continuity means that for every limit ordinal $\alpha < \omega_1$, the sequence $\langle p_\beta \mid \beta < \alpha \rangle$ converges to p_α.

Let $b \in \vec{K}$ with $b(0) = p_0$. For each $\alpha < \omega_1$, define $b_\alpha \in \vec{K}$ by letting $b_\alpha(0) = p_\alpha$ and for all $i < n$ with $i \neq 0$, $b_\alpha(i) = b(i)$.

Claim 1. For every club subset A of ω_1, there exists a club subset A' of A such that either $\langle (a \cdot b_\alpha)(i) \mid \alpha \in A' \rangle$ is strictly monotone or bounded.

Proof. Let A be an unbounded subset of ω_1. If there exists a club subset A' of A such that $\langle (a \cdot b_\alpha)(i) \mid \alpha \in A' \rangle$ is strictly monotone, then we are done. Suppose that there is no such A'.

Subclaim 1.1. There exists $\zeta < \omega_1$ such that for all $\alpha \in A$, there exists $\beta \in A_i \cap \zeta$ such that $(a \cdot b_\alpha)(i) \leq (a \cdot b_\beta)(i)$.

Proof. Suppose not. Then, for every $\zeta < \omega_1$, there exists $\alpha \in A$ such that for every $\beta \in A \cap \zeta$, $(a \cdot b_\beta)(i) < (a \cdot b_\alpha)(i)$. Hence, it is easy to build an uncountable subset A' of A such that $\langle (a \cdot b_\alpha)(i) \mid \alpha \in A' \rangle$ is strictly increasing. Note that for every limit point β of A', $\langle (a \cdot b_\alpha)(i) \mid \alpha \in A' \cap \beta \rangle$ converges to $(a \cdot b_\beta)(i)$. Thus, we may assume that A' is a club subset of A. This is a contradiction to the assumption. $\square$

For each $\beta < \zeta$, let B_β be the set of all $\alpha \in A$ such that $(a \cdot b_\alpha)(i) \leq (a \cdot b_\beta)(i)$. By the previous claim, we have $A = \bigcup_{\beta<\zeta} B_\beta$. Since ζ is countable, there exists $\beta < \zeta$ such that B_β is uncountable and hence unbounded in A. Let $B = B_\beta$. Then, B is an unbounded subset of A_i such that $\{ (a \cdot b_\alpha)(i) \mid \alpha \in B \}$ is bounded above.

Similarly, we can build an unbounded subset A' of B such that $\{ (a \cdot b_\alpha)(i) \mid \alpha \in A' \}$ is bounded below. Since $A' \subseteq B$, $\{ (a \cdot b_\alpha)(i) \mid \alpha \in A' \}$ is bounded. $\square$

By induction, we shall build a decreasing sequence $\langle A_i \mid i \leq n \rangle$ of unbounded subsets of ω_1 as follows. Let $A_0 = \omega_1$. Suppose that A_i has been defined. Then, by Claim 1, there exists a club subset A_{i+1} of A_i such that $\langle (a \cdot b_\alpha)(i) \mid \alpha \in A_{i+1} \rangle$ is either strictly monotone or bounded.

Consider A_n. Then, for every $i < n$, $\langle (a \cdot b_\alpha)(i) \mid \alpha \in A_n \rangle$ is either strictly monotone or bounded. Thus, by reindexing if necessary, without loss of generality, we may assume that for every $\langle (a \cdot b_\alpha)(i) \mid \alpha < \omega_1 \rangle$ is either strictly monotone or bounded.

Let s_0 be the set of all $i < n$ such that $\langle (a \cdot b_\alpha)(i) \mid \alpha < \omega_1 \rangle$ is strictly increasing, s_1 the set of all $i < n$ such that this set is strictly decreasing, and s_2 the set of all $i < n$ such that this set is bounded. For each $i \in s_2$, let $c(i)$ be a lower bound of $\langle (a \cdot b_\alpha)(i) \mid \alpha < \omega_1 \rangle$ and $d(i)$ be an upper bound of the same set.

Case 1. $s_0 = s_1 = \varnothing$.

Then, $\langle a \cdot b_\alpha \mid \alpha < \omega_1 \rangle$ is a sequence in $[c, d]$, which is compact. Thus, there exists $x \in [c, d]$ such that for every open neighborhood U of x and $\zeta < \omega_1$, there exists $\alpha < \omega_1$ such that $\zeta \leq \alpha$ and $a \cdot b_\alpha \in U$. So, since $\vec{K}$ is locally Euclidean, for every $\zeta < \omega_1$, there exists a countable increasing sequence $\langle \beta_m \mid m < \omega \rangle$ such that $\zeta \leq \beta_0$ and $\langle a \cdot b_{\beta_m} \mid m < \omega \rangle$ converges to x.

Thus, it is easy to build two increasing countable sequences $\langle \beta_m \mid m < \omega \rangle$ and $\langle \beta'_m \mid m < \omega \rangle$ such that for every $m < \omega$, $\beta_m < \beta'_0$, and both $\langle a \cdot b_{\beta_m} \mid m < \omega \rangle$ and $\langle a \cdot b_{\beta'_m} \mid m < \omega \rangle$ converge to x. Let $\beta = \sup_{m<\omega} \beta_m$ and $\beta' = \sup_{m<\omega} \beta'_m$. Then, we have both $a \cdot b_\beta$ and $a \cdot b_{\beta'}$ are equal to x. Since this semigroup is cancellative, we have $b_\beta = b_{\beta'}$. This is a contradiction.

Case 2. $s_0 \neq \varnothing$ or $s_1 \neq \varnothing$

For each $\beta < \omega_1$, define an open neighborhood W_β of $a \cdot b_{2\beta+1}$ by $x \in W_\beta$ if and only if

- for all $i \in s_0$, $(a \cdot b_{2\beta})(i) < x(i) < (a \cdot b_{2\beta+2})(i)$,
- for all $i \in s_1$, $(a \cdot b_{2\beta+2})(i) < x(i) < (a \cdot b_{2\beta})(i)$, and
- for all $i \in s_2$, $c(i) < x(i) < d(i)$.

Let $\{ U_m \mid m < \omega \}$ be a local basis at a, and $\beta < \omega_1$. Since $r_{b_{2\beta+1}}(a) = a \cdot b_{2\beta+1}$, $r_{b_{2\beta+1}}$ is continuous, and W_β is an open neighborhood of $a \cdot b_{2\beta+1}$, there exists $m_\beta < \omega$ such that $r_{b_{2\beta+1}}{}^{\to} U_{m_\beta} \subseteq W_\beta$.

So, there exists $m < \omega$ such that $A = \{ \beta < \omega_1 \mid m_\beta = m \}$ is uncountable. Let α be the limit point of A

Claim 2. For every $a' \in U_m$ and $i \in s_0$, $(a' \cdot b_\alpha)(i) = (a \cdot b_\alpha)(i)$.

Proof. Let $a' \in U_m$ and $i \in s_0$. Suppose $(a' \cdot b_\alpha)(i) \neq (a \cdot b_\alpha)(i)$. Since K_i is regular, there exists an open neighborhood W of $(a' \cdot b_\alpha)(i)$ such that $(a \cdot b_\alpha)(i) \notin \mathrm{cl}_{K_i}(W)$.

So, $K_i \setminus \mathrm{cl}_{K_i}(W)$ is an open neighborhood of $(a \cdot b_\alpha)(i)$. Thus, there exist $q_1, q_2 \in K_i$ such that $(a \cdot b_\alpha)(i) \in (q_1, q_2) \subseteq K_i \setminus \mathrm{cl}_{K_i}(W)$. Since l_a is continuous, $\langle a \cdot b_\beta \mid \beta < \alpha \rangle$ converges to $a \cdot b_\alpha$. So, there exists $\zeta < \alpha$ such that $(a \cdot b_\zeta)(i) \in (q_1, q_2)$. Thus, $[(a \cdot b_\zeta)(i), (a \cdot b_\alpha)(i)] \subseteq K_i \setminus \mathrm{cl}_{K_i}(W)$.

Since $l_{a'}$ is continuous, $\langle a' \cdot b_\beta \mid \beta < \alpha \rangle$ converges to $a' \cdot b_\alpha$. So, there exists $\eta < \alpha$ such that for all $\beta < \alpha$, if $\eta \leq \beta$, then $(a' \cdot b_\beta)(i) \in W$. Since α is a limit point of A, there exists $\beta \in A$ such that $\max\{ \zeta, \eta \} \leq \beta < \alpha$. Note $\max\{ \zeta, \eta \} \leq \beta < 2\beta + 1 < \alpha$. So, $(a' \cdot b_{2\beta+1})(i) \in W$. Meanwhile, we have $a' \in U_m = U_{m_\beta}$ and hence $a \cdot b_{2\beta+1} \in W_beta$. By the definition of W_β, $(a \cdot b_{2\beta})(i) < (a' \cdot b_{2\beta+1})(i) < (a \cdot b_{2\beta+2})(i)$. Since $\zeta \leq \beta \leq 2\beta$ and $2\beta + 2 < \alpha$, we have $(a \cdot b_\zeta)(i) < (a' \cdot b_{2\beta+1})(i) < (a \cdot b_\alpha)(i)$. This is a contradiction. $\square$

Similarly, we can prove the following claim.

Claim 3. For every $a' \in U_m$ and $i \in s_1$, $(a' \cdot b_\alpha)(i) = (a \cdot b_\alpha)(i)$.

Let $i \in s_0 \cup s_1$. r_{b_α} is a continuous injection from $\vec{K}$ into $\vec{K}$. By Theorem 2.1, $r_{b_\alpha}{}^{\to} U_m$ is an open neighborhood of $a \cdot b_\alpha$. So, there exists an $a' \in U_m$ such that $(a' \cdot b_\alpha)(i) \neq (a \cdot b_\alpha)(i)$. However, by the previous two claims, this is a contradiction.

□

In case that each K_i has no end points, we can easily obtain the following corollary.

Corollary 4.3. *Let n be a positive integer, and K_i a connected LOTS without end points for each $i < n$. If there is a cancellative topological semigroup on $\prod_{i<n} K_i$, then for every $i < n$, K_i is homeomorphic to $\mathbb{R}$. Thus, $\prod_{i<n} K_i$ is homeomorphic to $\mathbb{R}^n$.*

Proof. Let $i < n$. By Theorem 4.2, K_i is separable. By assumption, K_i has no endpoints. Since K_i is a connected LOTS, these properties imply that K_i is homeomorphic to $\mathbb{R}$. □

Recall that D. R. Brown and R. S. Houston proved the following theorem.

Theorem 4.4 (D. R. Brown and R. S. Houston [2])**.** *Let S be a cancellative semigroup on a connected and finite-dimensional, paracompact manifold. Then, there exists an analytic structure on S such that the semigroup operation is an analytic function from $S \times S$ to S.*

By Corollary 4.3 and Theorem 4.4, every cancellative topological semigroup S on the finite product of connected LOTS without end points has an analytic structure such that the semigroup operation is an analytic function from $S \times S$ to S.

5. Open problems

We focused on the case when the underlying space is the finite product of connected LOTS, which can be considered as a generalization of a Euclidean space. We may also consider generalizations of other manifolds, i.e., a topological space that is locally homeomorphic to the finite product of connected LOTS. For example, when S^1 is a circle and K is a connected LOTS, $S^1 \times K$ is such a space. We can prove that if K is nowhere separable, i.e., has no nonempty separable open subset, $S^1 \times K$ behaves very differently from $S^1 \times \mathbb{R}$. In particular, all local coordinate systems at any particular point must essentially coincide in their intersection.

Question 1. If there is a cancellative topological semigroup on a topological space that is locally homeomorphic to the finite product of connected LOTS, then is the underlying space separable? Locally separable?

Temporarily, we say that a topological space X has the IOD property if and only if every continuous injection from an open subset of X into X is an open map. We say that X has the productive IOD property if and only if for every positive integer n, X^n has the IOD property. In both this article and the semigroup argument in [7], the fact that the finite product of connected LOTS has the IOD property plays an important role. Also notice that such a space even has the productive IOD property. So, the following natural question arises.

Question 2. Let S be a topological space with the IOD property. If there is a cancellative topological semigroup on S, then is S separable? Locally separable? What if we require S to have the productive IOD property?

References

1. J. Aczél, *A short course on functional equations*, Theory and Decision Library. Series B: Mathematical and Statistical Methods, Based upon recent applications to the social and behavioral sciences, D. Reidel Publishing Co.,, Dordrecht, Netherlands, 1987. MR 875412
2. D. R. Brown and R. S. Houston, *Cancellative semigroups on manifolds*, Semigroup Forum **35** (1987), no. 3, 279–302. MR 900105
3. R. Craigen and Z. Páles, *The associativity equation revisited*, Aequationes Math. **37** (1989), no. 2-3, 306–312. MR 1004503
4. A. M. Gleason, *Groups without small subgroups*, Ann. of Math. (2) **56** (1952), 193–212. MR 49203
5. T. Ishiu, *Finite products of connected nowhere separable linearly ordered spaces*, Topology Appl. **300** (2021), Paper No. 107763, 21. MR 4291559
6. ———, *Brouwer's fixed-point theorem on the finite product of compact linearly ordered topological spaces*, Topology Appl. **362** (2025), Paper No. 109221, 14. MR 4854756
7. ———, *The invariance of domain theorem for the finite product of connected linearly ordered topological spaces*, Submitted, 2025.
8. Kenneth Kunen, *Set theory*, Studies in Logic (London), vol. 34, College Publications, London, London, U.K., 2011. MR 2905394
9. I. K. Lifanov, *The dimension of a product of ordered continua*, Dokl. Akad. Nauk SSSR **177** (1967), 778–781. MR 221480
10. D. Montgomery and L. Zippin, *Small subgroups of finite-dimensional groups*, Ann. of Math. (2) **56** (1952), 213–241. MR 49204
11. James R. Munkres, *Topology*, second ed., Prentice Hall, Inc., Upper Saddle River, NJ, 2000. MR 3728284