Injective continuous functions between the products of two connected nowhere real linearly ordered sets.

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Coordinate-wise functions

Definition

Let X_0, X_1, Y_0, Y_1 be sets. We say that a function $f : X_0 \times X_1 \to Y_0 \times Y_1$ is *coordinate-wise* if and only if there exist $i < 2$, $g_0 : X_i \to Y_0$, and $g_1 : X_{1-i} \to Y_1$ such that for every $\langle x_0, x_1 \rangle \in X_0 \times X_1$, $f(x_0, x_1) = \langle g_0(x_i), g_1(x_{1-i}) \rangle$.

Note that there are so many homeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$ that are not coordinate-wise

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The theorem of Eda and Kamijo

Here, $\mathrm{cf}(x)$ and $\mathrm{ci}(x)$ are the cofinality and coinitiality of x.

Theorem (K. Eda and R. Kamijo)

Let L be a connected linearly ordered set. If either $\mathrm{cf}(x)$ or $\mathrm{ci}(x)$ is uncountable for a dense set of x in L, then every homeomorphism $f : L^n \to L^n$ is coordinate-wise for every natural number n.

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Outline

Nowhere real linearly ordered sets

Definition

Let K be a linearly ordered set. We say that K is *nowhere real* if and only if no uncountable convex set of K is separable.

In other words, the closure of any countable set is nowhere dense.

For example, if a linearly ordered set satisfies the assumption of the Theorem of Eda and Kamijo, then it is nowhere real.

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Main Theorem

Theorem (I.)

Let K_0, K_1, L_0, L_1 be nowhere real connected linearly ordered sets. Then, every injective continuous function $f : K_0 \times K_1 \to L_0 \times L_1$ is coordinate-wise.

So, while it is restricted to dimension 2, this theorem improves the result of Eda and Kamijo in the following ways.

- Just being injective is sufficient.
- These four linearly ordered sets may be different (and they may be the same).
- Now, the completion of an Aronszajn line is covered.

In addition, the proof is totally different.

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Outline

Intermediate Value Theorem

Lemma

Let K and L be connected linearly ordered sets, and $g : K \to L$ a continuous function. Let $x_0 < x_1$ be both in K. If $z \in L$ is between $g(x_0)$ and $g(x_1)$, then there exists a $x_2 \in (x_0, x_1)$ such that $g(x_2) = z$.

Extreme Value Theorem

Lemma

Let K and L be connected linearly ordered sets and $g : K \to L$ a continuous function. Let $x_0 < x_1$ be both in K. Then, there exist maximum and minimum values of g on $[x_0, x_1]$.

Outline

Setting

Throughout this subsection, let K and L be nowhere real connected linerly ordered sets, $g : K \to L$ an injective continuous function, and M a countable elementary submodel of $H(\theta)$ with $K, L, f \in M$ for some sufficiently large regular cardinal θ.

Notations

Definition

Let $J(K, M)$ be the set of all $x \in K \setminus cl(K \cap M)$ with $\inf(K \cap M) < x < \sup(K \cap M)$.

For every $x \in K$, let

$$\begin{aligned}
\eta(K, M, x) &= \sup\{\, y \in K \cap M : y < x \,\} \\
\zeta(K, M, x) &= \inf\{\, y \in K \cap M : y > x \,\} \\
I(K, M, x) &= (\eta(K, M, x), \zeta(K, M, x)) \\
B(K, M, x) &= \{\, \eta(K, M, x), \zeta(K, M, x) \,\}
\end{aligned}$$

if they exist.

Note that if $x \in J(K, M)$, then both $\eta(K, M, x)$ and $\zeta(K, M, x)$ exist.

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Basic facts

Lemma

Let $x \in J(K, M)$. Then,

- *g is constant on $I(K, M, x)$ if and only if $g(\eta(K, M, x)) = g(\zeta(K, M, x))$.*
- *If $g(x) \in M$, then g is constant on $I(K, M, x)$.*
- *If $\eta(K, M, x) \notin M$ and $g(\eta(K, M, x)) \in M$, then g is constant on $I(K, M, x)$.*
- *If $g(x) \in J(L, M)$, then $g(\eta(K, M, x)) \in B(L, M, g(x))$.*

Of course, we can replace η by ζ in the previous lemma except the first item.

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Outline

2D to 1D

Now suppose K_0, K_1, L are nowhere real connected linearly ordered sets and consider $f : K_0 \times K_1 \to L$. Let M be a countable elementary submodel of $H(\theta)$ with $K_0, K_1, L, f \in M$.

If $y \in K_1 \cap M$, then the function $x \mapsto f(x, y)$ belongs to M. So, we can apply the result of the previous subsection. However, we can extend most of them to the case when y is a limit point of $K_1 \cap M$.

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Extended facts

Lemma

Let $\bar{x} \in J(K_0, M)$ *and* $\bar{y} \in \mathrm{cl}(K_1 \cap M)$*.*

- $x \mapsto f(x, \bar{y})$ *is constant on* $I(K_0, M, \bar{x})$ *if and only if* $f(\eta(K_0, M, \bar{x}), \bar{y}) = f(\zeta(K_0, M, \bar{x}), \bar{y})$*.*
- *If* $f(\bar{x}, \bar{y}) \in J(L, M)$*, then* $f(\eta(K_0, M, \bar{x}), \bar{y}) \in B(L, M, f(\bar{x}, \bar{y}))$*.*
- *If* $f(\eta(K_0, M, \bar{x}), \bar{y}) \in M$ *and* $\eta(K_0, M, \bar{x}) \notin M$*, then* $x \mapsto f(x, \bar{y})$ *is constant on* $I(K_0, M, \bar{x})$*.*

End point values are fixed near $\bar{y}$

Lemma

Let $\bar{x} \in J(K_0, M)$, and $\bar{y} \in K_1$ be a limit point of $K_1 \cap M$ from above. If $f(\bar{x}, \bar{y}) \in J(L, M)$, then there exists a $\bar{y}' \in K_1$ such that $\bar{y} < \bar{y}'$ and for every $y \in [\bar{y}, \bar{y}']$,

$$f(\eta(K_0, M, \bar{x}), y) = f(\eta(K_0, M, \bar{x}), \bar{y})$$
$$f(\zeta(K_0, M, \bar{x}), y) = f(\zeta(K_0, M, \bar{x}), \bar{y})$$

Facts about rectangles

Lemma

Let $\bar{x} \in J(K_0, M)$ and $\bar{y} \in J(K_1, M)$.

- $\max f''(I(K_0, M, \bar{x}) \times I(K_1, M, \bar{y})) = \max f''(B(K_0, M, \bar{x}) \times B(K_0, M, \bar{y}))$. *Similarly, for minimum.*
- *If $f(\bar{x}, \bar{y}) \in M$, then f is constant on $I(K_0, M, \bar{x}) \times I(K_1, M, \bar{y})$.*

Outline

2D to 2D

Let K_0, K_1, L_0, L_1 be nowhere real connected linearly ordered sets, and let $f : K_0 \times K_1 \to L_0 \times L_1$ be an injective continuous function. Let M be a countable elementary submodel of $H(\theta)$ with $K_0, K_1, L, f \in M$. Let $\langle g_0(x, y), g_1(x, y)\rangle = f(x, y)$.

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A rectangle is mapped to a rectangle

Lemma

Let $\bar{x} \in J(K_0, M)$ and $\bar{y} \in J(K_1, M)$. Let $a = \eta(K_0, M, \bar{x})$, $b = \zeta(K_0, M, \bar{x})$, $c = \eta(K_1, M, \bar{y})$, and $d = \zeta(K_1, M, \bar{y})$. Then, there exists $i < 2$ such that $g_i(a, c) = g_i(a, d)$, $g_i(b, c) = g_i(b, d)$, $g_{1-i}(a, c) = g_{1-i}(b, c)$, and $g_{1-i}(a, d) = g_{1-i}(b, d)$.

One coordinate is (almost) constant on a vertical line

Lemma

Let $\bar{x} \in J(K_0, M)$, $a = \eta(K_0, M, \bar{x})$, and $b = \zeta(K_0, M, \bar{x})$. Then, there exists $i < 2$ such that for every $y, y' \in (\inf(K_1 \cap M), \sup(K_1 \cap M))$, we have $g_i(a, y) = g_i(a, y')$ and $g_i(b, y) = g_i(b, y')$

Outline

One coordinate is constant on a vertical and horizontal line

Lemma

For every $x \in K_0$, there exists $i_X < 2$ such that $y \mapsto g_{i_x}(x, y)$ is constant on K_1.
Similarly, for every $y \in K_1$, there exists $j_y < 2$ such that $x \mapsto g_{j_y}(x, y)$ is constant on K_0.

Finally!

Lemma

There exist $i < 2$ such that for every $x \in K_0$, $i_x = i$ and for every $y \in K_1$, $j_y = 1 - i$. Hence, f is coordinate-wise.

This completes the proof of the main theorem.

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Open problems

- Higher dimension?
- Can we weaken “linearly ordered”?
- How about other things that can be done for $\mathbb{R}$?