THE INVARIANCE OF DOMAIN THEOREM FOR THE FINITE PRODUCT OF CONNECTED LINEARLY ORDERED TOPOLOGICAL SPACES

TETSUYA ISHIU

1. Introduction

In this article, we abbreviate a linearly ordered topological space as LOTS. The finite product of connected linearly without endpoints can be considered as a generalization of the Euclidean space $\mathbb{R}^n$. However, the behavior of such a product can be quite different. For example, L. B. Treybig showed following theorem.

Theorem 1.1 (Treybig [11])**.** *If X and Y are infinite Hausdorff spaces and $X \times Y$ is a continuous image of a compact LOTS, then both X and Y are metrizable.*

Notice that when X and Y satisfy the assumption of this theorem, then both X and Y are compact and metrizable and hence separable. Recall that G. Peano showed that $[0,1] \times [0,1]$ is a continuous image of $[0,1]$. Conversely, the previous theorem implies that if a compact LOTS K has this property, then K is separable and hence isomorphic to a subset of $\mathbb{R}$.

S. Mardešić [7] conjectured the following generalization of Theorem 1.1, which was proved by G. Martínez-Cervantes and G. Plebanek.

Theorem 1.2 (G. Martínez-Cervantes and G. Plebanek [8])**.** *Let d and s be positive integers. Let K_i be a compact LOTS for each $i < d$ and Z_j an infinite Hausdorff space for each $j < d + s$. If there exists a continuous surjective function from $\prod_{i<d} K_i$ onto $\prod_{j<d+s} Z_j$, then there exists at least $s+1$-many indexes $j < d + s$ such that Z_j is metrizable.*

Notice that Theorem 1.1 is a spacial case of this theorem when $d = s = 1$.

The author proved the following theorem, which demonstrates a significant difference between the Euclidean space and the finite product of connected nowhere separable LOTS. Here, a topological space X is said to be *nowhere separable* if and only if no nonempty open subset of X is separable. As usual in set theory, a natural number n is identified with the set of all natural numbers less than n, namely $n = \{ 0, 1, 2, \ldots, n-1 \}$. We shall regard an n-tuple as the function whose domain is n. Namely, $\langle x_0, x_1, \ldots, x_{n-1} \rangle$ is the function whose domain is n such that $x(i) = x_i$ for all $i < n$.

Date: January 1, 2026.

Theorem 1.3 (T. Ishiu [4]). *Let n be a positive integer. Let K_i and L_i be connected nowhere separable LOTS, and $f : \prod_{i<n} K_i \to \prod_{i<n} L_i$ be a continuous injection. Then, f is coordinate-wise, namely there exists a bijection $h : n \to n$ and a function $\tau_i : K_{h(i)} \to L_i$ for all $i < n$ such that for all $x \in \prod_{i<n} K_i$ and $k < n$,*

$$f(x)(i) = \tau_{h(i)}(x(h(i))).$$

Meanwhile, the author proved the following two theorems hold for the finite product of connected LOTS. Notice that Theorem 1.4 is analogous to Brouwer's fixed-point theorem while Theorem 1.5 is to the Poincaré-Miranda theorem.

Theorem 1.4 (L. E. J. Brouwer [2]). *Let n be a positive integer. Let K_i be a compact connected LOTS for each $i < n$. Then, every continuous function $f : \prod_{i<n} K_i \to \prod_{i<n} K_i$ has a fixed point, i.e., $x \in \prod_{i<n} K_i$ such that $f(x) = x$.*

Theorem 1.5 (C. Miranda [9]). *Let n be a positive integer. Let K_i and L_i be compact connected LOTS for each $i < n$. Define $\vec{K} = \prod_{i<n} K_i$, $\vec{L} = \prod_{i<n} L_i$, $a = \min \vec{K}$, and $b = \max \vec{K}$. Let $f : \vec{K} \to \vec{L}$ be a continuous function. Let $z \in \vec{L}$ be so that for every $i < n$ and $x \in \vec{K}$,*

- *if $x(i) = a(i)$, then $f(x)(i) < z$, and*
- *if $x(i) = b(i)$, then $f(x)(i) > z$*

Then $z \in \mathrm{ran}(f)$.

Note that the previous theorem works when $h : n \to n$ is a bijection and we assume if $x(i) = a(i)$, then $f(x)(h(i)) < z$ and if $x(i) = b(i)$, $f(x)(h(i)) > z$.

These theorems and their proofs suggest that the Euclidean spaces and other finite product of connected LOTS share some significant properties though their internal behavior may be quite different.

In this article, we shall prove the generalization of the following theorem of Brouwer, often called the invariance of domain theorem, to the finite product of connected LOTS.

Theorem 1.6 (Brouwer [1]). *For every positive integer n, every continuous injective function from $\mathbb{R}^n$ into $\mathbb{R}^n$ is an open map.*

Such a generalization is conjectured to be true in [6], where the following theorem was proved. Notice that when all K_i's and L_j's are isomorphic to $\mathbb{R}$, it is an easy corollary of Theorem 1.6

Theorem 1.7 (T. Ishiu [6]). *Let n and m be positive integers with $m < n$. Let K_i be a connected LOTS with $|K_i| \geq 2$ for each $i < n$ and L_j a connected LOTS for each $j < m$. Then, there exists no continuous injective function from a nonempty open subset of $\prod_{i<n} K_i$ into $\prod_{j<m} L_j$.*

The proof of the main theorem (Theorem 5.1) is done by the same strategy as in [5]. Namely, let $x \in \mathrm{dom}(f)$ and U an open neighborhood of x with $U \subseteq \mathrm{dom}\, f$. For a good enough countable elementary substructure M, we shall consider the condensations $\varphi(\prod_{i<n} K_i, M)$ and $\varphi(\prod_{j<n} L_j, M)$, and the function $\varphi(f, M) : \varphi(\prod_{i<n} K_i) \to \varphi(\prod_{j<n} L_j)$ that is correspondent to f. We shall observe that both $\varphi(\prod_{i<n} K_i, M)$ and $\varphi(\prod_{j<n} L_j, M)$ are homeomorphic to $\mathbb{R}^n$ and $\varphi(f, M)$ is injective. Thus, we can apply Theorem 1.6. If M is taken appropriately, we can show $f^{\to}U \subseteq \mathrm{ran}(f)$. To make this strategy work, we need the new lemmas, which are mainly proved in Section 4.

Then, as an application of the main theorem, we shall consider cancellative topological semigroups on the finite product of connected LOTS. In particular, we shall show that if such a semigroup exists, then the underlying space is locally Euclidean.

We also show that if every cancellative topological monoid on the finite product of connected LOTS is a Lie group. In this case, the underlying space must be the Euclidean space. So, the class of such monoids is limited. For example, recall that the unique abelian two-dimensional Lie group on $\mathbb{R}^2$ is $\mathbb{R}^2$ with the ordinary vector addition, and the unique nonabelian two-dimensional Lie group on $\mathbb{R}^2$ is $\mathrm{Aff}(1)$, the affine group of the line, up to isomorphism. Thus, the class of all cancellative topological monoids on the finite product of connected LOTS consists exactly of these two elements.

The structure of this article is as follows. In Section 2, we shall list definitions and lemmas about the finite product of connected LOTS. The most of them are taken from [5] and [6], but we shall also introduce a new notation C_e. In Section 3, we shall present definitions and lemmas about functions whose domain is the finite product of connected LOTS. In Section 4, we shall investigate the properties of the condenstations more deeply and prove lemmas that are critical in the proof of the main theorem. Section 5 is mostly devoted to the proof of the main theorem. In Section 6, we shall discuss an application of the main theorem and other results to topological semigroups.

2. The product of connected LOTS

2.1. **A single LOTS.** We would like to remind the readers some basic facts about LOTS. We will frequently use interval notations for a LOTS K. For example, if $a, b \in K$, then $[a, b)$ denotes the set of all elements $p \in K$ such that $a \leq p < b$.

Fact 2.1. Let K be a LOTS.

(i) If K is connected, then K is self-dense and has the least upper-bound property.
(ii) If K is separable and connected, then K is order-isomorphic to a convex subset of $\mathbb{R}$, i.e., one of $\{ 0 \}$, $[0, 1]$, $[0, 1)$, $(0, 1]$, and $(0, 1)$.

(iii) If K is connected and has both endpoints, then K is compact. In particular, any bounded closed interval $[a, b]$ is compact.
(iv) If K is compact, then K has both endpoints.

Now, we shall present the definitions and lemmas about the condensation defined by the author in [5] and [6]. In the following part of this subsection, we fix a compact connected LOTS K with $|K| \geq 2$, a regular cardinal $\theta \geq \theta_K$, and a countable elementary substructure M of $H(\theta)$ with $K \cap M$. Note that we assume K is compact to avoid dealing with elements p of K such that $p < \inf(K \cap M)$ or $p > \sup(K \cap M)$. However, the most of the arguments work without assuming the compactness.

The following definition is used to specify 'sufficiently large' cardinals.

Definition 2.2. Let $S_0, \ldots, S_{n-1}$ be any set. Let $\theta_{S_0,\ldots,S_{n-1}}$ be the least regular cardinal θ such that $\mathcal{P}(S_0 \cup S_1 \cup \cdots \cup S_{n-1}) \in H(\theta)$.

The following relation is defined in [5].

Definition 2.3. A relation $\sim_{K,M}$ on K is defined by for all $p, p' \in K$, $p \sim_{K,M} p'$ if and only if

- if $p \leq p'$, then $[p, p'] \cap M$ is finite, and
- if $p' < p$, then $[p', p] \cap M$ is finite.

When K and M are clear from the context, we omit them. We shall use this convention for all definitions defined by using K and M.

$\sim_{K,M}$ is shown to be an equivalence relation in [5]. We write $[p]_{K,M}$ to mean the $\sim_{K,M}$-equivalence class of p.

The following easy facts were proved in [5].

Fact 2.4. For every $p \in K$,

- $[p]$ is a closed convex subset of K and hence compact.
- $|[p] \cap M| \leq 1$.
- If $\min[p] \neq \min(K \cap M)$, then $\min[p]$ is a limit point of $K \cap M$ from below.
- If $\max[p] \neq \max(K \cap M)$, then $\max[p]$ is a limit point of $K \cap M$ from above.

Definition 2.5. Let $p \in K$. We shall define $C(K, M, p)$ to be the set of all $c \in K$ such that $c = \min[p]$, $c = \max[p]$, or $c \in [x] \cap M$.

The following technical lemmas were proved in [6].

Lemma 2.6. *Let $a, b \in K \cap M$ with $a < b$. Then, there exists $p \in (a, b) \cap M$ such that p is a limit point of $K \cap M$ from below. Similarly when we replace 'from below' by 'from above'.*

Lemma 2.7.

(i) Let $x \in K$ be a limit point of $K \cap M$ from below, and $a < x$. Then, there exists $a' \in K \cap M$ such that $a < a' < x$ and a' is a limit point of $K \cap M$ from below.

(ii) Let $x \in K$ be a limit point of $K \cap M$ from above, and $b > x$. Then, there exists $b' \in K \cap M$ such that $x < b' < b$ and b' is a limit point of $K \cap M$ from above.

Notice that if $a < p$ are elements of K and a is a limit point of $K \cap M$ from above, then $a < \min[p]$. Similarly, if $p < b$ are elements of K and b is a limit point of $K \cap M$ from below, then $\max[p] < b$. These facts are used frequently in the later sections.

The following notation is new in this paper.

Definition 2.8. Let $p \in K$. Define $C_e(x) = \{ \min[p], \max[p] \}$.

When $p \in K$, we shall denote the *cofinality of* p by $\mathrm{cf}(p)$ and the *coinitiality of* p as $\mathrm{ci}(p)$.

Fact 2.9. For every $p \in K$ and $c \in C(p)$, $c \in C_e(p)$ if and only if $c = \min K$, $c = \max K$, $\mathrm{ci}(c) = \omega$ or $\mathrm{cf}(c) = \omega$.

Proof. Let $c \in C(p)$. It is trivial when either $c = \min K$ or $C = \max K$. So, suppose $c \neq \min K$ and $c \neq \max K$.

If $c \in C_e(p)$, then $c = \min[p]$ or $c = \max[p]$. But we know that if $c = \min[p]$, then c is a limit point of $K \cap M$ from below and hence $\mathrm{cf}(p) = \omega$. Also, if $c = \max[p]$, then c is a limit point of $K \cap M$ from above and hence $\mathrm{ci}(p) = \omega$.

Suppose $c \notin C_e(p)$. Then, $\min[p] < c < \max[p]$ and $c \in M$. So, if $\mathrm{cf}(p) = \omega$, then $(\min[x], c) \cap M$ is infinite, which is a contradiction. Similarly, $ci(p) = \omega$ will derive a contradiction. □

By using this equivalence relation, we shall define the condensation of K.

Definition 2.10. Let $\varphi(K, M) = \{ [a] \mid a \in K \}$. We shall define a linear order on $\varphi(K, M)$ by $[a] \leq [b]$ if and only if $a \leq b$. This is well-defined as $[a]$ and $[b]$ are convex. The topology on $\varphi(K, M)$ is given by the order topology.

The following facts about $\varphi(K, M)$ were shown in [5].

Fact 2.11.

- $\varphi(K, M)$ is order-isomorphic to $[0, 1]$.
- The set $\{ [p] \mid p \in K \cap M \}$ is dense in $\varphi(K, M)$.

2.2. **The finite product.** In this subsection, we fix

- a positive integer n,
- a compact connected LOTS K_i for each $i < n$,
- a regular cardinal $\theta \geq \theta_{\langle K_i \mid i < n \rangle}$, and
- a countable elementary substructure M of $H(\theta)$ with $\langle K_i \mid i < n \rangle \in M$.

Let $\vec{K} = \prod_{i<n} K_i$. We shall use the following notations.

Definition 2.12. Let $x, y \in \vec{K}$. Define

(i) $x < y$ if and only if for every $i < n$, $x(i) < y(i)$

(ii) $x \leq y$ if and only if for every $i < n$, $x(i) \leq y(i)$,
(iii) $(x, y) = \{ z \in \vec{K} \mid x < z < y \}$, and
(iv) $[x, y] = \{ z \in \vec{K} \mid x \leq z \leq y \}$.

We shall use π_j to mean the projection to j-th coordinate, i.e.,

$$\pi_j(\langle x_0, x_1, \ldots, x_{n-1} \rangle) = x_j$$

The equivalence relation on $\vec{K}$ is defined as follows in [5].

Definition 2.13. Define a relation $\sim_{\vec{K},M}$ on $\vec{K}$ by $x \sim_{\vec{K},M} y$ if and only if for every $i < n$, $x(i) \sim_{K_i,M} y(i)$. Clearly, $\sim_{\vec{K},M}$ is an equivalence relation on $\vec{K}$. For every $x \in \vec{K}$, let $[x]_{\vec{K},M}$ be the $\sim_{\vec{K},M}$-equivalence class of x. We also define $C(\vec{K}, M, x) = \prod_{i<n} C(K_i, M, x(i))$. Let $\varphi(\vec{K}, M)$ be the set of all $\sim_{\vec{K},M}$-equivalence classes. It can be identified with $\prod_{i<n} \varphi(K_i, M)$. The topology on $\varphi(\vec{K}, M)$ is given as the product topology on $\prod_{i<n} \varphi(K_i, M)$.

The author [5] proved $C(x)$ has the following characterization.

Fact 2.14 (T. Ishiu [5])**.** For every $x \in \vec{K}$, $C(x) = [x] \cap \mathrm{cl}(\vec{K} \cap M)$.

We shall also define the vector version of C_e.

Definition 2.15. Let $x \in \vec{K}$. Define $C_e(\vec{K}, M, x) = \prod_{i<n} C_e(x(i))$.

The following notation help us describe subsets of $C(x)$.

Definition 2.16. Let $x \in \vec{K}$ and $s \subseteq n$. Define $C(\vec{K}, M, x, s)$ to be the set of all $c \in C(\vec{K}, M, c_0)$ such that $c \restriction s = x \restriction s$.

For example, if $x \in \vec{K}$, $i_0 < n$, and $x(i_0) = \min[x(i_0)]$, then $C(x, \{ i_0 \})$ is the set of all $c \in C(x)$ such that $c(i_0) = \min[x(i_0)]$.

Definition 2.17. Let $x \in \vec{K}$ be so that $[x] \subseteq \mathrm{dom}(f)$. Let $s(x)$ be the set of all $i < n$ such that $|[x(i)]| > 1$.

So, $i \in n \setminus s(x)$ is equivalent to $|[x(i)]| = 1$ and hence $[x(i)] = \{ x(i) \}$. Notice that for every $i < n$, if $i \in s(x)$, then $|C_e(x)| = 2$, and if $i \notin s(x)$, then $|C_e(x)| = 1$. So,

$$|C_e(x)| = \left| \prod_{i<n} C_e(x) \right| = \prod_{i<n} |C_e(x(i))| = 2^{|s(x)|}$$

The following lemmas are used to ensure $[x] \subseteq \mathrm{dom}(f)$ in a typical situation when f is a function on an open subset of $\vec{K}$.

Lemma 2.18. *Let $x_0 \in \vec{K} \cap M$ and $U \in M$ an open neighborhood of x_0. Suppose that for every $i < n$, $\min K_i < x_0(i) < \max K_i$. Then $[x_0] \subseteq U$.*

Proof. Since U is open and for every $i < n$, $\min K_i < x_0(i) < \max K_i$, there exists $a, b \in \vec{K}$ such that $x_0 \in (a, b) \subseteq U$. Since $x_0, U \in M$, without loss of generality, we may assume $a, b \in M$. Since $x_0 \in (a, b)$, for every $i < n$, we have $a(i) < x_0(i) < b(i)$. Since $x_0, a, b \in M$ and K_i is self-dense, there exist $a'(i), b'(i) \in K_i \cap M$ such that $a(i) < a'(i) < x_0(i) < b'(i) < b(i)$. Then, for every $i < n$, clearly $a'(i) \leq \min[x_0(i)]$ and $\max[x_0(i)] \leq b'(i)$. So, we have $[x_0] \subseteq [a', b'] \subseteq (a, b) \subseteq U$. □

Lemma 2.19. *Let $x_0 \in \vec{K}$ and U an open set such that $[x_0] \subseteq U$ and for every $x \in U$ and $i < n$, $\min K_i < x < \max K_i$. Then, there exist $a, b \in \vec{K} \cap M$ such that $x_0 \in (a, b) \subseteq U$ and for all $x \in (a, b)$, $[x] \subseteq (a, b)$.*

Proof. Since $[x_0] \subseteq U$ and U is open, for each $x \in [x_0]$, there exist $a_x, b_x \in \vec{K}$ such that $x \in (a_x, b_x) \subseteq U$. It is easy to see that $[x_0]$ is compact. So, there exists a finite subset F of $[x_0]$ such that $[x_0] \subseteq \bigcup_{x \in F} (a_x, b_x)$.

Claim 1. For every $i < n$, there exists $x \in F$ such that $a_x(i) < \min[x_0(i)]$.

Proof. Let $c \in \vec{K}$ be defined by $c(i) = \min[x_0(i)]$ for all $i < n$. Then, $c \in [x_0]$. So, there exists $x \in F$ such that $c \in (a_x, b_x)$. Thus, for every $i < n$, $a_x(i) < c(i) = \min[x_0(i)]$. □

So,

$$\max \{ a_x(i) \mid x \in F \wedge a_x(i) < \min[x_0(i)] \} < \min[x_0(i)]$$

By Lemma 2.7, since $\min[x_0(i)]$ is a limit point of $K_i \cap M$ from below, there exists $a(i) \in K_i \cap M$ such that $a(i)$ is a limit point of $K_i \cap M$ from above and

$$\max \{ a_x(i) \mid x \in F \wedge a_x(i) < \min[x_0(i)] \} < a(i) < \min[x_0(i)]$$

Similarly, there exists $b(i) \in K_i \cap M$ such that $b(i)$ is a limit point of $K_i \cap M$ from above and

$$\max[x_0(i)] < b(i) < \min \{ b_x(i) \mid x \in F \wedge \max[x_0(i)] < b_x(i) \}$$

Claim 2. $x_0 \in (a, b)$.

Proof. By the definition of a and b, for every $i < n$,

$$a(i) < \min[x_0(i)] \leq x_0(i) \leq \max[x_0(i)] < b(i)$$

□

Claim 3. For every $x \in (a, b)$, $[x] \subseteq (a, b)$

Proof. Let $x \in (a, b)$ and $i < n$. Then, $a(i) < x(i) < b(i)$. Since $a(i)$ is a limit point of $K_i \cap M$ from above, we have $a(i) < \min[x(i)]$. Similarly, we have $\max[x(i)] < b(i)$. So, $[x] \subseteq (a, b)$. □

Claim 4. $(a, b) \subseteq U$.

Proof. Let $x \in (a, b)$. We shall define $x' \in [x_0]$ as follows. For each $i < n$, if $x(i) \in [x_0(i)]$, then $x'(i) = x(i)$. If $x(i) < \min[x_0(i)]$, then let $x'(i) = \min[x_0(i)]$. If $x(i) > \max[x_0(i)]$, then let $x'(i) = \max[x_0(i)]$. Then, $x' \in [x_0]$. So, there exists $y \in F$ such that $x' \in (a_y, b_y)$.

We shall show that $x \in (a_y, b_y)$. Let $i < n$. If $x(i) \in [x_0(i)]$, then $x(i) = x'(i) \in (a_y(i), b_y(i))$. If $x(i) < \min[x_0(i)]$, then since $x \in (a, b)$, we have $a(i) < x(i)$. Since $x'(i) = \min[x_0(i)] \in (a_y(i), b_y(i))$, we have $a_y(i) < \min[x_0(i)]$. By the definition of a, we have $a_y(i) < a(i)$. So, $a_y(i) < x(i)$. Similarly, if $x(i) > \max[x_0(i)]$, then $x(i) < a_y(i)$.

Therefore, $x \in (a_y, b_y) \subseteq U$. □

□

3. A function from the product

3.1. Into a single LOTS.

In this section, we shall review the definitions and results on a function from the product of connected LOTS into a single LOTS in [5] and [6], and then prove several new lemmas that are essential to prove the main theorem.

In this section, we fix

- a positive integer n,
- a compact connected LOTS K_i for each $i < n$,
- a compact connected LOTS L,
- a continuous function g from a nonempty open subset of $\prod_{i<n} K_i$ into L such that for all $x \in \mathrm{dom}(g)$ and $i < n$, $\min K_i < x(i) < \max K_i$,
- a regular cardinal $\theta \geq \theta_g$, and
- a countable elementary substructure M of $H(\theta)$ with $g \in M$.

As in the last subsection, let $\vec{K} = \prod_{i<n} K_i$.

The following facts were proved in [5].

Fact 3.1. Let $x \in \vec{K}$. Then,

(i) $g^{\to}[x] \subseteq [g(x)]$.
(ii) $\min g^{\to}[x] = \min g^{\to}C(x)$ and $\max g^{\to}[x] = \max g^{\to}C(x)$.

By Fact 3.1 (i), we can see that the following definition is well-defined.

Definition 3.2. $\varphi(g, M)$ is defined to be the function from $\{\, [x] \mid x \in \mathrm{dom}(g) \,\} \subseteq \varphi(\vec{K}, M)$ into $\varphi(L, M)$ such that

$$\varphi(g, M)([x]) = [g(x)]$$

The following fact was proved in [5].

Fact 3.3. $\varphi(g, M)$ is continuous.

The following lemma is a generalization of a lemma proved in [6].

Lemma 3.4. *Let $c_0 \in \vec{K}$ and $s \subseteq n$ be so that $[c_0] \subseteq \mathrm{dom}(g)$. Define $s_0 = \{\, i \in s \mid c_0(i) = \min[c_0(i)] \,\}$ and $s_1 = \{\, i \in s \mid c_0(i) = \max[c_0(i)] \,\}$. Suppose*

- $s \neq \varnothing$,
- $[c_0] \subseteq \mathrm{dom}(g)$,
- $s = s_0 \cup s_1$ (*i.e., for every* $i \in s$, $c_0(i) \in C_e(c_0(i))$), *and*
- g *is not constant on* $\{\, x \in [c_0] \mid x \restriction s = c_0 \restriction s \,\}$

Then, there exists $a, b \in \vec{K}$ *such that* $c_0 \in (a, b) \subseteq \mathrm{dom}(g)$ *and for all* $x \in (a, b)$, $[x] \subseteq \mathrm{dom}(g)$ *and if*

- $x \restriction (n \setminus s) = c_0 \restriction (n \setminus s)$,
- *for all* $i \in s_0$, $x(i) < c_0(i)$, *and*
- *for all* $i \in s_1$, $x(i) > c_0(i)$,

then for all $d \in C(x)$, *we have* $g(d) = g((c_0 \restriction s) \cup (d \restriction (n \setminus s)))$.

Proof. Suppose that c_0, s, s_0, s_1 are as in the assumption. By Lemma 2.19, there exists $\tilde{a}, \tilde{b} \in \vec{K}$ such that $c_0 \in (\tilde{a}, \tilde{b}) \subseteq \mathrm{dom}(g)$ and for all $x \in (\tilde{a}, \tilde{b})$, $[x] \subseteq (\tilde{a}, \tilde{b}) \subseteq \mathrm{dom}(g)$.

Since g is not constant on $\{\, x \in [c_0] \mid x \restriction s = c_0 \restriction s \,\}$, its image under g is a connected subset of L with at least two points. So, it has infinitely many points. Since $g^{\to}[c_0] \cap \mathrm{Cl}(L \cap M) \subseteq [g(c_0)] \cap \mathrm{Cl}(L \cap M)$ is finite, there exists $x_0 \in [c_0]$ such that $x_0 \restriction s = c_0 \restriction s$ and $g(x_0) \notin \mathrm{Cl}(L \cap M)$. Since g is continuous, there exists $a', b' \in \vec{K}$ such that $x_0 \in (a', b') \subseteq (\tilde{a}, \tilde{b})$ and $g^{\to}(a', b') \cap \mathrm{Cl}(L \cap M) = \varnothing$. So, $g^{\to}(a', b')$ is a connected subset of L that is disjoint from $\mathrm{Cl}(L \cap M)$. Thus, for all $x \in (a', b')$, since both $g(x_0)$ and $g(x)$ belong to $g^{\to}(a', b')$, we have $g(x) \sim g(x_0)$. Since $x_0 \in [c_0]$, we have $g(x_0) \sim g(c_0)$. So, $g(x) \sim g(c_0)$.

Let $c \in C(c_0, s)$. Since $[g(c)] \cap \mathrm{Cl}(L \cap M)$ is finite, there exist $a_c, b_c \in \vec{K}$ such that $c \in (a_c, b_c) \subseteq (\tilde{a}, \tilde{b})$ and $g^{\to}(a_c, b_c) \cap \mathrm{Cl}(L \cap M) \subseteq \{\, g(c) \,\}$.

Define $a, b \in \vec{K}$ as follows. For all $i \in n \setminus s$, let $a(i) = a_{c_0}(i)$ and $b(i) = b_{c_0}(i)$. Let $i \in s$. Then, $c_0(i) = x_0(i) \in (a'(i), b'(i))$. Also, for all $c \in C(c_0, s)$, $c_0(i) = c(i) \in (a_c(i), b_c(i))$. Let $a(i) \in K_i$ be so that

- $a(i) > a'(i)$,
- $a(i) > a_c(i)$ for all $c \in C(c_0, s)$,
- $a(i) < c_0(i)$, and
- if $i \in s_0$, then $a(i)$ is a limit point of $K_i \cap M$ from above.

The last condition can be attained since if $i \in s_0$, then $c_0(i)$ is a limit point of $K_i \cap M$ from below. Let $b(i) \in K_i$ be so that

- $b(i) < b'(i)$,
- $b(i) < b_c(i)$ for all $c \in C(c_0, s)$,
- $b(i) > c_0(i)$, and
- if $i \in s_1$, then $b(i)$ is a limit point of $K_i \cap M$ from below.

We shall show that this satisfies the conclusion. Let $x \in (a, b)$. Then, we have $(a, b) \subseteq (a_{c_0}, b_{c_0}) \subseteq (\tilde{a}, \tilde{b})$. So, $[x] \subseteq (\tilde{a}, \tilde{b}) \subseteq \mathrm{dom}(g)$.

Suppose $x \restriction (n \setminus s) = c_0 \restriction (n \setminus s)$, $x(i) < c_0(i)$ for all $i \in s_0$, and $x(i) > c_0(i)$ for all $i \in s_1$. Let $d \in C(x)$. Define $c = (c_0 \restriction s) \cup (d \restriction (n \setminus s))$. It suffices to show that $g(d) = g(c)$.

Claim 1. $c \in C(c_0, s)$.

Proof. Let $i \in s$. By the definition, $c(i) = c_0(i)$.

Let $i \in n \setminus s$. Then, $c(i) = d(i) \in C(x(i))$. Since $x(i) = c_0(i)$, we have $C(x(i)) = C(c_0(i))$. So, $c(i) \in C(c_0(i))$. □

Claim 2. $d \in (a_c, d_c)$.

Proof. Let $i < n$. Suppose $i \in n \setminus s$. Then, $d(i) = c(i) \in (a_c(i), b_c(i))$.

Suppose $i \in s$. Then, $d(i) \sim x(i)$. Since $x \in (a, b)$, we have $a(i) < x(i) < b(i)$. If $i \in s_0$, then $x(i) < c_0(i)$. Since $c_0(i)$ is a limit point of $K_i \cap M$ from below and $a(i)$ is a limit point of $K_i \cap M$ from above,

$$a(i) < \min[x(i)] \leq x(i) \leq \max[x(i)] < c_0(i) < b(i)$$

If $i \in s_1$, similarly we have

$$a(i) < c_0(i) < \min[x(i)] \leq x(i) \leq \max[x(i)] < b(i)$$

In either case, we have $a(i) < \min[x(i)] \leq \max[x(i)] < b(i)$. Since $d(i) \sim x(i)$, $\min[x(i)] \leq d(i) \leq \max[x(i)]$. So, $a(i) < d(i) < b(i)$. □

Since $d \in C(x)$, we have $g(d) \in C(g(x)) \subseteq \mathrm{Cl}(L \cap M)$. Therefore,

$$g(d) \in g^{\rightarrow}(a_c, b_c) \cap \mathrm{Cl}(L \cap M) \subseteq \{\, g(c) \,\}$$

So, $g(d) = g(c)$. □

Lemma 3.5. *Let $c_0 \in \vec{K}$ and $s \subseteq n$ be so that $[c_0] \subseteq \mathrm{dom}(g)$. Set $q = g(c_0)$. Let $s_0 = \{\, i \in s \mid c_0(i) = \min[c_0(i)] \,\}$ and $s_1 = \{\, i \in s \mid c_0(i) = \max[c_0(i)] \,\}$. Suppose that*

- *$s \neq \varnothing$,*
- *$[c_0] \subseteq \mathrm{dom}(g)$,*
- *$s = s_0 \cup s_1$, and*
- *for every $c \in C(c_0, s)$, $g(c) = q$.*

Then, for all $y \in [c_0]$ with $y \restriction s = c_0 \restriction s$, $g(y) = q$.

Proof. Suppose that c_0, s, q satisfy the assumption and define s_0 and s_1 as in the statement. Suppose that the conclusion is false. Then, g is not constant on $\{\, y \in [c_0] \mid y \restriction s = c_0 \restriction s \,\}$. Thus, by Lemma 3.4, there exists $a, b \in \vec{K}$ that satisfies the conclusion. Also, there exists $y_0 \in [c_0]$ such that $y_0 \restriction s = c_0 \restriction s$ and $g(y_0) \neq q$. Without loss of generality, we may assume $g(y_0) \notin \mathrm{Cl}(L \cap M)$. So, there exist $a', b' \in \vec{K}$ such that $y_0 \in (a', b')$, $q \notin g^{\rightarrow}(a', b')$, and $g^{\rightarrow}(a', b') \cap \mathrm{Cl}(L \cap M) = \varnothing$. Let $y_1 \in (a', b')$ be so that

- $y_1 \restriction (n \setminus s) = y_0 \restriction (n \setminus s)$,
- for all $i \in s_0$, $\max\{\, a(i), a'(i) \,\} < y_1(i) < y_0(i)$, and
- for all $i \in s_1$, $y_0(i) < y_1(i) < \min\{\, b(i), b'(i) \,\}$.

Define $x \in \vec{K}$ by $x = (y_1 \restriction s) \cup (c_0 \restriction (n \setminus s))$.

Claim 1. $x \in (a, b)$

Proof. Let $i < n$. Suppose $i \in n \setminus s$. Then, by the definition, $x(i) = c_0(i) \in (a(i), b(i))$.

Suppose $i \in s_0$. Then, $a(i) < y_1(i) < y_0(i) = c_0(i) < b(i)$. Suppose $i \in s_1$. Then, $a(i) < c_0(i) = y_0(i) < y_1(i) < b(i)$. Since $x(i) = y_1(i)$, in both cases, we have $x(i) \in (a(i), b(i))$. □

Claim 2. $g^{\rightarrow}[x] = \{ q \}$

Proof. By definition, we have

- $x \restriction (n \setminus s) = c_0 \restriction (n \setminus s)$,
- for all $i \in s_0$, $x(i) = y_1(i) < y_0(i) = c_0(i)$, and
- for all $i \in s_1$, $x(i) = y_1(i) > y_0(i) = c_0(i)$.

Thus, by the definition of a and b, for all $d \in C(x)$, we have $g(d) = g(c)$ where $c = (c_0 \restriction s) \cup (d \restriction (n \setminus s))$. However, we have $c \in C(c_0, s)$ and hence $g(c) = q$. Therefore, for every $d \in C(x)$, we have $g(d) = q$. By Fact 3.1 (ii), it implies that $g^{\rightarrow}[x] = \{ q \}$. □

Claim 3. $y_1 \in [x]$.

Proof. Let $i < n$. Suppose $i \in s$. Then, $y_1(i) = x(i) \in [x(i)]$. Suppose $i \notin s$. Then, $y_1(i) = y_0(i) \in [c_0(i)]$. Since $x(i) = c_0(i)$, we have $[c_0(i)] = [x(i)]$. Therefore, $y_1(i) \in [x(i)]$. □

Thus, $g(y_1) = q$. However, since $q \notin g^{\rightarrow}(a', b')$ and $y_1 \in (a', b')$, we have $g(y_1) \neq q$. This is a contradiction. □

Lemma 3.6. *Let $c_0 \in \vec{K}$ be so that $[c_0] \subseteq \mathrm{dom}(g)$ and for every $i < n$, $c_0(i) \in C_e(c_0(i))$. Suppose that g is not constant on $[c_0]$. Then, there exist $a, b \in \vec{K}$ such that $c_0 \in (a, b) \subseteq \mathrm{dom}(g)$ and for all $x \in (a, b)$ with $x \restriction s(c_0) = c_0 \restriction s(c_0)$, $g(x) = g(c_0)$.*

Proof. If $s(c_0) = \varnothing$, then $[c_0]$ is a singleton and hence g is constant on $[c_0]$. Thus, we have $s(c_0) \neq \varnothing$.

By Lemma 2.19, there exist $\tilde{a}, \tilde{b} \in \vec{K}$ such that $c_0 \in (\tilde{a}, \tilde{b}) \subseteq \mathrm{dom}(g)$ and for all $x \in (\tilde{a}, \tilde{b})$, $[x] \subseteq (\tilde{a}, \tilde{b}) \subseteq \mathrm{dom}(g)$.

Since g is not constant on $[c_0]$, there exists $y \in [c_0]$ such that $g(y) \notin \mathrm{Cl}(L \cap M)$. So, there exists $a_0, b_0 \in \vec{K}$ such that $y \in (a_0, b_0) \subseteq (\tilde{a}, \tilde{b})$ and $g^{\rightarrow}(a_0, b_0) \cap \mathrm{Cl}(L \cap M) = \varnothing$. Thus, for every $x \in (a_0, y_0)$, $g(x) \sim g(y)$.

Since $C(g(c_0))$ is finite, there exist $a_1, b_1 \in \vec{K}$ such that $c_0 \in (a_1, b_1) \subseteq (\tilde{a}, \tilde{b})$ and $g^{\rightarrow}(a_1, b_1) \cap C(g(c_0)) \subseteq \{ g(c_0) \}$.

Define $a, b \in \vec{K}$ as follows. For every $i \in s$, let $a(i) = a_1(i)$ and $b(i) = b_1(i)$. Notice that $a_1(i) < c_0(i) < b_1(i)$. Let $i \notin s$. By the definition of s, $|[c_0(i)]| = 1$ and hence $[c_0(i)] = \{ c_0(i) \}$. Since $y(i) \in [c_0(i)]$, we have $y(i) = c_0(i)$. Since $a_0(i) < y(i) < b_0(i)$, we have $a_0(i) < c_0(i) < b_0(i)$. Let $a(i)$ be a limit point of $K_i \cap M$ from above such that $\max\{ a_0(i), a_1(i) \} < a(i) < c_0(i)$ and let $b(i)$ be a limit point of $K_i \cap M$ from below such that $c_0(i) < b(i) < \min\{ b_0(i), b_1(i) \}$. Then, it is easy to see $c_0 \in (a, b)$.

Claim 1. For all $x \in (a, b)$ and $c \in C(x)$, if $c \restriction s = c_0 \restriction s$, then $g(c) = g(c_0)$.

Proof. Let x and c be as in the assumption.

Subclaim 1.1. $c \in (a, b)$.

Proof. Let $i < n$. Suppose $i \in s$. Then, $c(i) = c_0(i)$ by the assumption. By the definition of a and b, we have $a(i) < c_0(i) < b(i)$. Hence $c(i) \in (a(i), b(i))$.

Suppose $i \notin s$. Since $x \in (a, b)$, we have $a(i) < x(i) < b(i)$. Since $a(i)$ is a limit point of $K_i \cap M$ from below, we have $a(i) < \min[x(i)]$. Since $b(i)$ is a limit point of $K_i \cap M$ from above, we have $\max[(x(i))] < b(i)$. Since $c(i) \in C(x(i))$, we have $\min[x(i)] \leq c(i) \leq \max[x(i)]$. So, $a(i) < c(i) < b(i)$ ane hence $c(i) \in (a(i), b(i))$. □

Define $y' = (y \restriction s) \cup (c \restriction (n \setminus s))$.

Subclaim 1.2. $y' \in (a_0, b_0)$

Proof. Let $i < n$. Suppose $i \in s$. Then, $y'(i) = y(i) \in (a_0(i), b_0(i))$.

Suppose $i \notin s$. Then, $y'(i) = c(i)$. Since $c \in (a, b)$, $c(i) \in (a(i), b(i)) \subseteq (a_0(i), b_0(i))$. □

So, $g(y') \sim g(y) \sim g(c_0)$.

Subclaim 1.3. $c \in C(y')$

Proof. Let $i < n$. Suppose $i \in s$. Then, $c(i) = c_0(i)$. We have $y'(i) = y(i) \in [c_0(i)]$. Thus, $c(i) \in [y'(i)]$. Since $c(i) \in C(x(i)) \subseteq \mathrm{Cl}(K_i \cap M)$, we have $c(i) \in [y'(i)] \cap \mathrm{Cl}(K_i \cap M) = C(y'(i))$.

Suppose $i \notin s$. Then, $c(i) = y'(i) \in [y'(i)]$. So, $c(i) \in [y'(i)] \cap \mathrm{Cl}(K_i \cap M) = C(y'(i))$. □

So, $g(c) \in C(g(y')) = C(g(c_0))$. Thus,

$$g(c) \in g^{\rightarrow}(a, b) \cap C(g(c_0)) \subseteq g^{\rightarrow}(a_1, b_1) \cap C(g(c_0)) \subseteq \{ g(c_0) \}$$

So, $g(c) = g(c_0)$. □

For all $x \in (a, b)$, since $(a, b) \subseteq (\tilde{a}, \tilde{b})$, we have $[x] \subseteq (\tilde{a}, \tilde{b}) \subseteq \mathrm{dom}(g)$. By Lemma 3.5, the previous claim implies that if $x \restriction s(c_0) = c_0 \restriction s(c_0)$, then $g(x) = g(c_0)$. □

3.2. **Into the product of connected LOTS.** We shall extend it to functions from the product of connected LOTS into the product of connected LOTS. In the remainder of this paper, when f is a function whose range is a set of tuples, we shall write f_j to mean $\pi_j \circ f$. So, for every $x \in \mathrm{dom}(f)$, we have $f_j(x) = f(x)(j)$.

Definition 3.7. Suppose

- n is a positive integer,
- for each $i < n$, K_i and L_i are compact connected LOTS,

- f is a continuous function from a nonempty open subset of $\prod_{i<n} K_i$ into $\prod_{i<n} L_i$ such that for all $x \in \mathrm{dom}(f)$ and $i < n$, $\min K_i < x(i) < \max K_i$,
- θ is a regular cardinal with $\theta \geq \theta_f$, and
- M is a countable elementary substructure of $H(\theta)$ with $f \in M$.

We shall define the function $\varphi(f, M)$ from $\{ [x] \mid x \in \mathrm{dom}(f) \}$ into $\varphi(\vec{L}, M)$ by for every $x \in \mathrm{dom}(f)$,

$$\varphi(f, M)(x)(j) = \varphi(f_j, M)(x)$$

where f_j is the j-th coordinate function of f, namely $f_j = \pi_j \circ f$.

Since $\varphi(f_j, M)$ is continuous, $\varphi(f, M)$ is continuous.

4. A box is mapped onto a box.

The following notation will be used in the following sections.

Definition 4.1. Let n be a positive integer. Let K_i be a connected LOTS for each $i < n$. Let f be a continuous function on a nonempty open subset of $\prod_{i<n} K_i$. For every $x \in \vec{K}$ with $[x] \subseteq \mathrm{dom}(f)$, let $t_f(x)$ be the set of all $j < n$ such that f_j is not constant on $[x]$. When f is clear from the context, we shall write $t(x)$ to mean $t_f(x)$.

For each positive integer n, let $(*)_n$ denote the following statement.

- Suppose that
  - K_i is a compact connected LOTS with $|K_i| \geq 2$ for each $i < n$,
  - L_j is a connected LOTS for each $j < n$,
  - $\vec{K} = \prod_{i<n} K_i$, and $\vec{L} = \prod_{j<n} L_j$,
  - f is a continuous injective function from an nonempty open subset of $\vec{K}$ into $\vec{L}$,
  - for all $x \in \mathrm{dom}(f)$ and $i < n$, $\min K_i < x(i) < \max K_i$,

  Then, f is an open map.

In this section, we shall assume

- n is a positive integer,
- K_i, L_j, f are as in the assumption of $(*)_n$,
- θ is a regular cardinal $\theta \geq \theta_{\vec{K},\vec{L},f}$,
- M is a countable elementary substructure of $H(\theta)$ such that $\vec{K}, \vec{L}, f \in M$, and
- $(*)_m$ holds for all $m < n$.

Lemma 4.2. *For every $x \in \vec{K}$ with $[x] \subseteq \mathrm{dom}(f)$, if $s(x) = \varnothing$, then $t(x) = \varnothing$.*

Proof. Suppose that $s(x) = \varnothing$. Then $[x] = \{ x \}$. Thus, for every $j < n$, f_j is constant on $[x]$. So, $t(x) = \varnothing$. □

Lemma 4.3. *Let $c_0 \in \vec{K}$ such that $c_0 \in C_e(c_0)$ and $[c_0] \subseteq \mathrm{dom}(f)$. Then, there exist $a, b \in \vec{K}$ such that $c_0 \in (a, b) \subseteq \mathrm{dom}(f)$ and* *(i) for every $x \in (a,b)$, if $x \restriction s(c_0) = c_0 \restriction s(c_0)$, then $f(x) \restriction t = f(c_0) \restriction t$,*

(ii) if we define a function $g : \prod_{i \in n \setminus s}(a(i), b(i)) \to \prod_{j \in n \setminus t} L_j$ by

$$g(x) = f(x \cup (c_0 \restriction s(c_0))) \restriction (n \setminus t(c_0)),$$

then g is a continuous injection.

Proof. First, we shall focus on (i). If $s(c_0) = \varnothing$, then by Lemma 4.2, $t(c_0) = \varnothing$. Thus, $f(x) \restriction t(c_0) = f(c_0) \restriction t(c_0)$ vacuously holds for every x. So, from now on, suppose $s(c_0) \neq \varnothing$.

By Lemma 2.19, there exists $\tilde{a}, \tilde{b} \in \vec{K}$ such that $c_0 \in (\tilde{a}, \tilde{b}) \subseteq \mathrm{dom}(f)$ and for every $x \in (\tilde{a}, \tilde{b})$, $[x] \subseteq (\tilde{a}, \tilde{b})$.

Let $j \in t(c_0)$. Then, f_j is not constant on $[c_0]$. So, by Lemma 3.6, there exist $a_j, b_j \in \vec{K}$ such that $c_0 \in (a_j, b_j) \subseteq (\tilde{a}, \tilde{b})$ and for all $x \in (a_j, b_j)$ with $x \restriction s(c_0) = c_0 \restriction s(c_0)$, $f_j(x) = f_j(c_0)$.

For each $i < n$, let $a(i) = \max\{\, a_j(i) \mid j \in t(c_0) \,\}$ and $b(i) = \min\{\, b_j(i) \mid j \in t(c_0) \,\}$. Then, for all $x \in (a,b)$ with $x \restriction s(c_0) = c_0 \restriction s(c_0)$, we have $f(x) \restriction t(c_0) = f(c_0) \restriction t(c_0)$. So, (i) is proved.

Now, we shall show (ii). Define g as in the statement. Clearly, g is continuous. To show that g is injective, let x and x' be distinct elements of $\prod_{i \in n \setminus s(c_0)}(a(i), b(i))$. Then, since $x \cup (c_0 \restriction s(c_0)) \neq x' \cup (c_0 \restriction s(c_0))$ and f is injective, we have $f(x \cup (c_0 \restriction s(c_0))) \neq f(x' \cup (c_0 \restriction s(c_0)))$. By (i), $f(x \cup (c_0 \restriction s(c_0))) \restriction t = f(c_0) \restriction t = f(x' \cup (c_0 \restriction s(c_0))) \restriction t$. Thus, $g(x) = f(x \cup (c_0 \restriction s(c_0))) \restriction (n \setminus t(c_0)) \neq f(x' \cup (c_0 \restriction s(c_0))) \restriction (n \setminus t(c_0) = g(x')$. $\square$

Lemma 4.4. *Let $x_0 \in \mathrm{dom}(f)$ such that $[x_0] \subseteq \mathrm{dom}(f)$. Then, $|s(x_0)| = |t(x_0)|$.*

Proof. If $s(x_0) = \varnothing$, by Lemma 4.2, $t(x_0) = \varnothing$. So, $|s(x_0)| = |t(x_0)| = 0$. Thus, we shall assume $s(x_0) \neq \varnothing$ from now on.

First, we shall show $|s(x_0)| \leq |t(x_0)|$. Define a function $g : \prod_{i \in s}[x_0(i)] \to \prod_{j \in t} L_j$ by

$$g(x) = f(x \cup (x_0 \restriction (n \setminus s(x_0)))) \restriction t$$

Clearly, g is continuous.

We shall show that g is injective. Suppose that x and x' are distinct elements of $\prod_{i \in s(x_0)}[x_0(i)]$. So, $x \cup (x_0 \restriction (n \setminus s(x_0))) \neq x' \cup (x_0 \restriction (n \setminus s(x_0)))$. Thus, $f(x \cup (x_0 \restriction (n \setminus s(x_0)))) \neq f(x' \cup (x_0 \restriction (n \setminus s(x_0))))$. So, there exists $j < n$ such that $f_j(x \cup (x_0 \restriction (n \setminus s(x_0)))) \neq f_j(x' \cup (x_0 \restriction (n \setminus s(x_0))))$. If $j \notin t(x_0)$, then f_j is constant on $[x_0]$, which is a contradiction. Therefore, $j \in t(x_0)$. Hence, $g(x) \neq g(x')$. By Theorem 1.7, $|s(x_0)| \leq |t(x_0)|$.

Then, we shall show $|s(x_0)| \geq |t(x_0)|$. Let $c_0 \in C(x_0)$ be defined by $c_0(i) = \min[x_0(i)]$. Let a and b be as in the conclusion of Lemma 4.3. Define a function g as in (ii) of Lemma 4.3. Then, g is a continuous injection from $\prod_{i\in n\setminus s(x_0)}(a(i),b(i))$ into $\prod_{j\in n\setminus t(x_0)} L_j$. By Theorem 1.7, we have $|n\setminus s(x_0)| \le |n\setminus t(x_0)|$. So, $|s(x_0)| \ge |t(x_0)|$. □

Lemma 4.5. *Let $c_0 \in \vec{K}$ such that $c_0 \in C_e(c_0)$ and $[c_0] \subseteq \mathrm{dom}(f)$. Suppose $s(c_0) \neq \varnothing$. Then, there exists an open neighborhood W of $f(c_0)$ such that for every $z \in W$, if $z \upharpoonright t(c_0) = f(c_0) \upharpoonright t(c_0)$, then there exists $x \in \vec{K}$ such that $x \upharpoonright s(c_0) = c_0 \upharpoonright s(c_0)$ and $f(x) = z$.*

Proof. Let a, b, g be as in the conclusion of Lemma 4.3 (ii). Then, g is a continuous injection from $\prod_{i\in n\setminus s(c_0)}(a(i),b(i))$ into $\prod_{j\in n\setminus t(c_0)} L_j$. By Lemma 4.4, $|s(c_0)| = |t(c_0)|$. So, we have $|n\setminus s(c_0)| = |n\setminus t(c_0)|$. Thus, we can apply $(*)_{n-|s(c_0)|}$ to show that g is an open map.

So, $\mathrm{ran}(g)$ is an open neighborhood of $g(c_0)$. Let W be the set of all $z \in \vec{L}$ such that $z \upharpoonright (n\setminus t(c_0)) \in \mathrm{ran}(g)$. We shall show that this W works. Clearly W is an open subset of $\vec{L}$. We have

$$\begin{aligned} g(c_0 \upharpoonright (n\setminus s(c_0))) &= f((c_0 \upharpoonright (n\setminus s(c_0))) \cup (c_0 \upharpoonright s(c_0))) \upharpoonright (n\setminus t(c_0)) \\ &= f(c_0) \upharpoonright (n\setminus t(c_0)) \end{aligned}$$

So, $f(c_0) \in W$. Let $z \in W$ be so that $z \upharpoonright t(c_0) = f(c_0) \upharpoonright t(c_0)$. By the definition of W, $z \upharpoonright (n\setminus t(c_0)) \in \mathrm{ran}(g)$, so there exists $y \in \prod_{i\in n\setminus s(c_0)}(a(i),b(i))$ such that $g(y) = z \upharpoonright (n\setminus t(c_0))$. By the definition of g, it implies

$$f(y \cup (c_0 \upharpoonright s(c_0))) \upharpoonright (n\setminus t(c_0)) = z \upharpoonright (n\setminus t(c_0))$$

By Lemma 4.3 (i), $f(y \cup (c_0 \upharpoonright s(c_0))) \upharpoonright t(c_0) = f(c_0) \upharpoonright t(c_0) = z \upharpoonright t(c_0)$. Therefore, $f(y \cup (c_0 \upharpoonright s(c_0))) = z$. □

Lemma 4.6. *Let $x_0 \in \mathrm{dom}(f)$ such that $[x_0] \subseteq \mathrm{dom}(f)$. Then, for each $i_0 \in s(x_0)$, there exists a unique $j_0 \in t(x_0)$ such that f_{j_0} is constant on $\{\, x \in [x_0] \mid x(i_0) = \min[x_0(i_0)] \,\}$.*

Proof. If $s(x_0) = \varnothing$, then $[x_0]$ is a singleton. So, f_{j_0} is trivially constant on $\{\, x \in [x_0] \mid x(i_0) = \min[x_0(i_0)] \,\}$. Thus, suppose $s(c_0) \neq \varnothing$.

First, we shall show the existence. Let $i_0 \in s(c_0)$. By way of contradiction, suppose that for every $j \in t(x_0)$, f_j is not constant on $\{\, x \in [x_0] \mid x(i_0) = \min[x_0(i_0)] \,\}$. Let $c_0 \in C(x_0)$ be defined by $c_0(i) = \min[x_0(i)]$ for all $i < n$. By Lemma 4.5, there exists an open neighborhood W of $f(c_0)$ such that for every $z \in W$, if $z \upharpoonright t(c_0) = f(c_0) \upharpoonright t(c_0)$, then there exists $y \in \vec{K}$ such that $y \upharpoonright s(c_0) = c_0 \upharpoonright s(c_0)$ and $f(y) = z$. Since f is continuous, there exists $a, b \in \vec{K}$ such that $c_0 \in (a,b) \subseteq \mathrm{dom}(f)$ and $f^{\rightarrow}(a,b) \subseteq W$.

Let $j \in t(c_0)$. Since f_j is not constant on $\{\, x \in [x_0] \mid x(i_0) = \min[x_0(i_0)] \,\}$, by Lemma 3.4 (applied with $s = \{\, i_0 \,\}$), there exists $a_j, b_j \in \vec{K}$ such that $c_0 \in (a_j, b_j) \subseteq \mathrm{dom}(g)$, and for all $x \in (a_j, b_j)$, $[x] \subseteq \mathrm{dom}(f)$ and if $x \upharpoonright (n\setminus\{\, i_0 \,\}) = c_0 \upharpoonright (n\setminus\{\, i_0 \,\})$ and $x(i_0) < c_0(i_0)$, then for all $d \in C(x)$, we have $f_j(d) = f_j((c_0 \upharpoonright \{\, i_0 \,\}) \cup (d \upharpoonright (n\setminus\{\, i_0 \,\})))$.

Define $a', b' \in \vec{K}$ by for each $i < n$, let $a'(i) = \max(\{\, a(i) \,\} \cup \{\, a_j(i) \mid j \in t(c_0) \,\})$ and $b'(i) = \min(\{\, b(i) \,\} \cup \{\, b_j(i) \mid j \in t(c_0) \,\})$.

Let $x \in (a', b')$ be so that $x \restriction (n \setminus \{ i_0 \}) = c_0 \restriction (n \setminus \{ i_0 \})$. Define $d \in (a', b')$ by $d \restriction (n \setminus \{ i_0 \}) = c_0 \restriction (n \setminus \{ i_0 \})$ and $d(i_0) = \max[x(i_0)]$. Then, $d \in C(x)$. Since $x \in (a', b') \subseteq \bigcap_{j \in t}(a_j, b_j)$ and $x \restriction (n \setminus \{ i_0 \}) = c_0 \restriction (n \setminus \{ i_0 \})$, we have

$$f(d) \restriction t(c_0) = f((c_0 \restriction \{ i_0 \}) \cup (d \restriction (n \setminus \{ i_0 \}))) \restriction t(c_0) = f(c_0) \restriction t(c_0)$$

Since $d \in (a', b') \subseteq (a, b)$, we have $f(d) \in W$. By the definition of W, since $f(d) \restriction t = f(c_0) \restriction t$, there exists $y \in \vec{K}$ such that $y \restriction s(c_0) = c_0 \restriction s(c_0)$ and $f(y) = f(d)$. Since f is injective, we have $y = d$. This is a contradiction since $d(i_0) < c(i_0) = y(i_0)$. So, there exists $j_0 \in t(c_0)$ such that f_{j_0} is constant on $\{ x \in [x_0] \mid x(i_0) = \min[x_0(i_0)] \}$.

Now, we shall show the uniqueness. Suppose that j_0 and j_1 are distinct elements of $t(c_0)$ and both f_{j_0} and f_{j_1} are constant on $\{ x \in [x_0] \mid x(i_0) = \min[x_0(i_0)] \}$. Define a function $g' : \prod_{i \in n \setminus \{ i_0 \}}[x_0(i)] \to \prod_{j \in n \setminus \{ j_0, j_1 \}} L_j$ by

$$g'(x) = f(x \cup \{ \langle i_0, \min[c_0(i_0)] \rangle \}) \restriction (n \setminus \{ j_0, j_1 \}).$$

It is easy to see that g' is a continuous injection. This is a contradiction to Theorem 1.7.

□

Lemma 4.7. *For all $a, b \in \vec{K} \cap M$, if $a < b$ and $(a, b) \subseteq \mathrm{dom}(f)$, there exists $c \in (a, b) \cap M$ such that $c \in C_e(c)$ and $f(c) \in C_e(f(c))$.*

Proof. In M, we shall build a sequence $\langle a_k, b_k, c_k \mid k < \omega \rangle$ such that for every $k < \omega$, $a_k < b_k$, $c_k \in (a_k, b_k)$ and $(a_{k+1}, b_{k+1}) \subseteq (a_k, b_k)$.

Let $a_0 = a$ and $b_0 = b$. Suppose that a_k and b_k are defined. Let $c_k \in (a_k, b_k)$ be so that for all $m < k$ and $i < n$, $c_k(i) \neq c_m(i)$.

We shall build a sequence $\langle a_{k,j}, b_{k,j} \mid j \leq n \rangle$ such that for every $j \leq n$, $a_{k,j} < b_{k,j}$. Let $a_{k,0} = a_k$ and $b_{k,0} = b_k$. Suppose that $a_{k,j}$ and $b_{k,j}$ are defined. If f_j is constant on $(a_{k,j}, b_{k,j})$, we can derive a contradiction by Theorem 1.7. So, f_j is not constant on $(a_{k,j}, b_{k,j})$. Thus, there exists $c_{k,j}$ such that $f_j(c_{k,j}) \neq f_j(c_k)$. Since f_j is continuous, there exists $a_{k,j+1}, b_{k,j+1} \in \vec{K}$ such that $c_{k,j} \in (a_{k,j+1}, b_{k,j+1})$, $(a_{k,j+1}, b_{k,j+1}) \subseteq (a_{k,j}, b_{k,j})$, and for every $x \in (a_{k,j+1}, b_{k,j+1})$, $f_j(x) \neq f_j(c_k)$. Let $a_{k+1} = a_{k,n}$ and $b_{k+1} = b_{k,n}$

This construction can be carried out in M. So, without loss of generality, we may assume $\langle a_k, b_k, c_k \mid k < \omega \rangle \in M$.

By construction, for every $m < k < \omega$ and $j < n$, $f_j(c_m) \notin f_j{}^{\rightarrow}(a_k, b_k)$. So, $f_j(c_m) \neq f_j(c_k)$. Thus, there exists an increasing sequence $\langle k_l \mid l < \omega \rangle$ such that for every $i < n$, $\langle c_{k_l}(i) \mid l < \omega \rangle$ is strctly monotone and for every $j < n$, $\langle f_j(c_{k_l}) \mid l < \omega \rangle$ is strictly monotone.

Let c be the limit point of $\langle c_{k_l} \mid l < \omega \rangle$. Since $\langle c_{k_l}(i) \mid l < \omega \rangle$ is strictly monotone for all $i < n$, this c is unique. Since $\langle c_{k_l} \mid l < \omega \rangle \in M$, $c \in M \cap \mathrm{Cl}(\vec{K} \cap M)$. So, $c \in C_e(c)$. Also, for each $j < n$, $\langle f_j(c_{k_l}) \mid l < \omega \rangle$ belongs to M and is strictly monotone, $f(c) \in \mathrm{Cl}(\vec{L} \cap M)$. Thus, $f(c) \in C_e(f(c))$. □

Lemma 4.8. *For every $c \in \mathrm{dom}(f)$, if $c \in C_e(c)$, then $f(c) \in C_e(f(c))$.*

Proof. We already know that $f(c) \in C(f(c))$. So, it suffices to show that $f(c)$ is a limit point of $\vec{L} \cap M$. Let W be an open neighborhood of $f(c)$. Then, $f^{\leftarrow}W$ is an open neighborhood of c. Thus, there exist $a, b \in \vec{K}$ such that $c \in (a, b) \subseteq f^{\leftarrow}W$. We shall define $a', b' \in \vec{K} \cap M$ as follows. Let $i < n$. If $c(i) = \min[c(i)]$, then since $\min[c(i)]$ is a limit point of $K_i \cap M$ from below, there exist $a'(i), b'(i) \in K_i \cap M$ such that $a(i) < a'(i) < b'(i) < c(i)$. If $c(i) = \max[c(i)]$, then since $\max[c(i)]$ is a limit point of $K_i \cap M$ from above, there exist $a'(i), b'(i) \in K_i \cap M$ such that $c(i) < a'(i) < b'(i) < b(i)$.

By Lemma 4.7, there exists $c' \in (a', b') \cap M$ such that $f(c') \in C_e(f(c)) \subseteq \mathrm{Cl}(\vec{L} \cap M)$. Since $(a', b') \subseteq (a, b)$, c is a limit point of $\vec{L} \cap M$. $\square$

Lemma 4.9. *Let $x_0 \in \mathrm{dom}(f)$ such that $[x_0] \subseteq \mathrm{dom}(f)$. Suppose $h : s(x_0) \to t(x_0)$ is a function such that for every $i \in s(x_0)$, $f_{h(i)}$ is constant on $\{\, x \in [x_0] \mid x(i) = \min[x_0(i)] \,\}$. Then, h is a bijection.*

(i) for every $i \in s$, $f_{h(i)}$ is constant on $\{\, x \in [x_0] \mid x(i) = \max[x_0(i)] \,\}$,
(ii) for every $i \in s$ and $c, c' \in C_e(x_0)$, if $c(i) \neq c'(i)$, then $f_{h(i)}(c) \neq f_{h(i)}(c')$, and
(iii) h is a bijection.

Proof. Note that by Lemma 4.6, the function h exists uniquely.

Let $c_0 \in C_e(x_0)$ be defined by $c_0(i) = \min[x_0(i)]$ for all $i < n$. Let $d_0 = f(c_0)$. By Lemma 4.8, $d_0 \in C_e(f(x_0))$. For each $j < n$, let $d_1(j) \in C_e(f_j(x_0))$ such that $d_0 \neq d_1$.

For (i), let $i \in s(c_0)$. Then, for every $c \in C_e(x_0)$, if $c(i) = \min[x_0(i)]$, we have $f_{h(i)}(c) = f_{h(i)}(c_0) = d_0(h(i))$. Notice

$$\begin{aligned} |\{\, c \in C_e(x_0) \mid c(i) = \min[x_0(i)] \,\}| &= 2^{|s(c_0)|-1} = 2^{|t(c_0)|-1} \\ &= |\{\, d \in C_e(f(x_0)) \mid d(i) = d_0(h(i)) \,\}| \end{aligned}$$

Since f is injective,

$$f^{\rightarrow}\{\, c \in C_e(x_0) \mid c(i) = \min[x_0(i)] \,\} = \{\, d \in C_e(f(x_0)) \mid d(i) = d_0(h(i)) \,\}$$

So, for every $c \in C_e(x_0)$, if $c(i) = \max[x_0(i)]$, then $f_{h(i)}(c) \neq d_0(h(i))$. Since $f_{h(i)}(c) \in C_e(f_{h(i)}(x_0))$, we have $f_{h(i)}(c) = d_1(h(i))$. By Lemma 3.5, for every $x \in [x_0]$, if $x(i) = \max[x_0(i)]$, then $f_{h(i)}(x) = d_1(h(i))$.

For (ii), let $i \in s$ and $c, c' \in C_e(x_0)$ be so that $c(i) \neq c'(i)$. Since $c(i), c'(i) \in C_e(x_0(i))$, we may assume $c(i) = \min[x_0(i)]$ and $c'(i) = \max[x_0(i)]$. In the previous paragraph, we showed $f_{h(i)}(c) = d_0(h(i)) \neq d_1(h(i)) = f_{h(i)}(c')$.

For (iii), since $|s(c_0)| = |t(c_0)|$, it suffices to show that h is injective. Let i_0, i_1 be distinct elements of s such that $h(i_0) = h(i_1)$. Let $c_1 \in C_e(x_0)$ be defined by $c_1 \restriction (n \setminus \{\, i_0 \,\}) = c_0 \restriction (n \setminus \{\, i_0 \,\})$ and $c_1(i_0) = \max[x_0(i_0)]$. Then, since $c_1(i_1) = c_0(i_1)$, we have $f_{h(i_1)}(c_1) = f_{h(i_1)}(c_0)$. Meanwhile, by (ii), since $c_1(i_0) \neq c_0(i_0)$, we have $f_{h(i_0)}(c_1) \neq f_{h(i_0)}(c_0)$. However, since $h(i_0) = h(i_1)$, this is a contradiction. $\square$

Lemma 4.10. *Let $x_0 \in \operatorname{dom} f$ such that $[x_0] \subseteq \operatorname{dom}(f)$. Then, $f^{\rightarrow}[x_0] = [f(x_0)]$.*

Proof. We already know $f^{\rightarrow}[x_0] \subseteq [f(x_0)]$. So, it suffices to show $[f(x_0)] \subseteq f^{\rightarrow}[x_0]$. Define $c_0 \in C_e(x_0)$ by $c_0(i) = \min[x_0(i)]$ for every $i < n$ and $c_1 \in C_e(x_0)$ by $c_1(i) = \max[x_0(i)]$. Let h be the function defined in Lemma 4.9.

First, we shall show that for every $z \in [f(x_0)]$, if $\min[f_j(x_0)] < z(j) < \max[f_j(x_0)]$ for every $j \in t(x_0)$, then $z \in f^{\rightarrow}[x_0]$. For each $j \in t(x_0)$, since h is bijective, there exists $i \in s(x_0)$ such that $h(i) = j$. So, for every $x \in [x_0]$, if $x(i) = \min[x_0(i)]$, then $f_j(x) = f_j(c_0)$. Also, if $x(i) = \max[x_0(i)]$, then $f_j(x) = f_j(c_1)$. Notice that $z(j)$ is strictly between $f_j(c_0)$ and $f_j(c_1)$. By Theorem 1.5, there exists $x \in [x_0]$ such that $f(x) = z$.

Let $z \in [f(x_0)]$. Clearly, for every open neighborhood U of z, there exists $z' \in [f(x_0)]$ such that $\min[f_j(x_0)] < z'(j) < \max[f_j(x_0)]$ for every $j \in t(x_0)$. We have shown $z' \in f^{\rightarrow}[x_0]$. So, $z \in \operatorname{Cl}(f^{\rightarrow}[x_0]) = f^{\rightarrow}[x_0]$. □

5. Main Theorem

Theorem 5.1. *Let n be a positive integer, K_i a connected LOTS with no end points and $|K_i| \geq 2$ for each $i < n$, L_j a connected LOTS for each $j < n$, and f a continuous injective function from a nonempty open subset of $\prod_{i<n} K_i$ into $\prod_{j<n} L_j$. Then, f is an open map.*

Proof. First, we shall show that it suffices to show that $(*)_n$ suffices. Suppose that n, K_i, L_j, f be as in the assumption and suppose $(*)_n$ holds. Notice that we cannot apply $(*)_n$ directly since K_i is not compact. We shall show that f is an open map. Let U be a nonempty open subset of $\operatorname{dom}(f)$. To show that $f^{\rightarrow}U$ is open, let $z \in f^{\rightarrow}U$. Then, there exists $x \in U$ such that $f(x) = z$. Let θ be a regular cardinal such that $\theta \geq \theta_f$, and M a countable elementary substructure of $H(\theta)$ with $\langle K_i \mid i < n\rangle, \langle L_j \mid j < n\rangle, f, U, z, x \in M$. For each $i < n$, since K_i has no end points, there exist $a(i), b(i) \in K_i \cap M$ such that $a(i) < x(i) < b(i)$. Let $K_i' = [a(i), b(i)]$ for every $i < n$. Define $f' = f \restriction \prod_{i<n}(a(i), b(i))$. Then, K_i', L_j, f' satisfy the assumption of $(*)_n$. Thus, f' is an open map. So, $(f')^{\rightarrow}(U \cap \prod_{i<n}(a(i), b(i)))$ is an open neighborhood of $f(x) = z$. Trivially, we have $(f')^{\rightarrow}(U \cap \prod_{i<n}(a(i), b(i))) \subseteq f^{\rightarrow}U$. Therefore, $f^{\rightarrow}U$ is open.

Now, we shall prove $(*)_n$ for all positive integer n by induction. $(*)_1$ is trivial. Let n be a positive integer and suppose that $(*)_m$ holds for all $m < n$. Let K_i, L_j, f be as in the assumption of $(*)_n$. Let $\vec{K} = \prod_{i<n} K_i$ and $\vec{L} = \prod_{i<n} L_i$.

Let U be a nonempty subset of $\operatorname{dom}(f)$. We shall show that $f^{\rightarrow}U$ is open in $\vec{L}$. Let $x_0 \in U$. It suffices to find a basic open neighborhood W of $f(x_0)$ such that $W \subseteq f^{\rightarrow}U$. Let θ be a regular cardinal such that $\theta \geq \theta_{\vec{K},\vec{L},f}$. Let M be a countable elementary submodel of $H(\theta)$ with $\vec{K}, \vec{L}, f, U, x_0 \in M$.

By Lemma 2.18, $[x_0] \subseteq U$. By Lemma 2.19, there exist $a_0, b_0 \in \vec{K} \cap M$ such that $x_0 \in (a_0, b_0) \subseteq U$ and for every $x \in (a_0, b_0)$, $[x] \subseteq U$.

Claim 1. $\{\, [x] \mid x \in (a_0, b_0) \,\} = ([a_0], [b_0])$. In particular, $\{\, [x] \mid x \in (a_0, b_0) \,\}$ is open in $\varphi(\vec{K}, M)$.

Proof. We shall show $\{\, [x] \mid x \in (a_0, b_0) \,\} = ([a_0], [b_0])$. To see $\{\, [x] \mid x \in (a_0, b_0) \,\} \subseteq ([a_0], [b_0])$, let $x \in (a_0, b_0)$. By the definition of a_0, b_0, we have $[x] \subseteq (a_0, b_0)$. So, for every $i < n$, $a_0 < \min[x(i)]$ and $\max[x(i)] < b_0(i)$. Thus, $[a_0] < [x] < [b_0]$.

To see $([a_0], [b_0]) \subseteq \{\, [x] \mid x \in (a_0, b_0) \,\}$, let $x \in \vec{K}$ be so that $[x] \in ([a_0], [b_0])$. Then, for every $i < n$, $a_0(i) < x(i) < b_0(i)$ and hence $x \in (a_0, b_0)$. $\square$

Claim 2. $\varphi(f, M)$ is injective on $([a_0], [b_0])$.

Proof. Let $x, x' \in \vec{K}$ be so that $[x], [x'] \in ([a_0], [b_0])$. Then, $x, x' \in (a_0, b_0)$. Suppose $\varphi(f, M)([x]) = \varphi(f, M)([x'])$. Then, $[f(x)] = [f(x')]$. By Lemma 4.10, we have $f^{\to}[x] = [f(x)]$ and $f^{\to}[x'] = [f(x')]$. Thus, $f^{\to}[x] = f^{\to}[x']$. So, $f(x') \in f^{\to}[x]$. Thus, there exists $x'' \in [x]$ such that $f(x') = f(x'')$. Since f is injective, $x' = x'' \sim x$. So, $[x] = [x']$. $\square$

Both $([a_0], [b_0])$ and $\varphi(\vec{L}, M)$ are isomorphic to $\mathbb{R}^n$. By Theorem 1.6, $\varphi(f, M)$ is an open map. So, $\varphi(f, M)^{\to}([a_0], [b_0])$ is open . Recall $x_0 \in (a_0, b_0)$ and hence $[f(x_0)] \in \varphi(f, M)^{\to}([a_0], [b_0])$. So, there exist $d_0, d_1 \in \vec{L}$ such that $[f(x_0)] \in ([d_0], [d_1]) \subseteq \varphi(f, M)^{\to}([a_0], [b_0])$. Define $d_0', d_1' \in \vec{L}$ by $d_0'(i) = \max[d_0(i)]$ and $d_1'(i) = \min[d_1(i)]$ for all $i < n$. Since $[f(x_0)] \in ([d_0], [d_1])$, clearly $f(x_0) \in (d_0', d_1')$.

Claim 3. $(d_0', d_1') \subseteq f^{\to}(a_0, b_0)$.

Proof. Let $z \in (d_0', d_1')$. Then, $[z] \in ([d_0], [d_1]) \subseteq \varphi(f, M)^{\to}([a_0], [b_0])$. So, there exists $x \in \vec{K}$ such that $[x] \in ([a_0], [b_0])$ and $\varphi(f, M)([x]) = [z]$. Since $[x] \in ([a_0], [b_0])$, we have $x \in (a_0, b_0)$. By Lemma 4.10, we have $f^{\to}[x] = [f(x)] = [z]$. So, there exists $x' \in [x]$ such that $f(x') = z$. $\square$

Therefore, there exists a basic open neighborhood (d_0', d_1') of $f(x_0)$ such that $(d_0', d_1') \subseteq f^{\to}(a_0, b_0) \subseteq f^{\to}U$. Thus, $f^{\to}U$ is open. $\square$

We shall also prove the following corollary, which will be used in Section 6.

Corollary 5.2. *Let n be a positive integer. For each $i < n$, let K_i be a connected LOTS with no end points and $|K_i| \geq 2$, and L_i a connected LOTS. Let f be a continuous injection from an open subset of $\prod_{i<n} K_i$ into $\prod_{j<n} L_j$. Then, for all $x_0 \in \mathrm{dom}(f)$,*

$$\begin{aligned}&|\{\, i < n \mid x_0(i) \textit{ has a separable open neighborhood} \}| \\ &\qquad\qquad = |\{\, j < n \mid f(x_0)(j) \textit{ has a separable open neighborhood} \}|\end{aligned}$$

Proof. Let $\vec{K} = \prod_{i<n} K_i$ and $\vec{L} = \prod_{i<n} L_i$. Let $x_0 \in \mathrm{dom}(f)$. Let s' be the set of all $i < n$ such that $x_0(i)$ has a separable open neighborhood and t' the set of all $j < n$ such that $f(x_0)(j)$ has a separable open neighborhood. So, the conclusion can be expressed as $|s'| = |t'|$

For each $i \in s'$, let U_i be a separable open neighborhood of $x_0(i)$. For each $i \in n \setminus s'$, let $U_i = K_i$. Then, $\mathrm{dom}(f) \cap \prod_{i<n} U_i$ is an open neighborhood of x_0. So, there exists $\langle U_i' \mid i < n \rangle$ such that $x_0(i) \in U_i' \subseteq U_i$ and $\prod_{i<n} U_i' \subseteq \mathrm{dom}(f)$. Let $U' = \prod_{i<n} U_i'$

By Theorem 5.1, $f^{\rightarrow}U'$ is an open subset of $\vec{L}$. Thus, there exist $a, b \in \vec{L}$ such that $f(x_0) \in (a, b) \subseteq f^{\rightarrow}U'$. Let θ be a regular cardinal such that $\theta \geq \theta_{\vec{K},\vec{L},f}$, and M a countable elementary substructure of $H(\theta)$ such that $\vec{K}, \vec{L}, f, x_0, U', a, b \in M$.

We shall define $z \in (a, b)$ as follows. For each $j \in t'$, let $z(j) = f(x_0)(j)$. For each $j \notin t'$, by the definition of t', $f(x_0)(j)$ has no separable open neighborhood. In particular, $(a(j), b(j))$ is not separable. Thus, $(a(j), b(j)) \setminus \mathrm{Cl}(L_j \cap M) \neq \varnothing$. Let $z(j) \in (a(j), b(j)) \setminus \mathrm{Cl}(L_j \cap M)$.

Since $z \in (a, b) \subseteq f^{\rightarrow}U'$, there exists $x \in U'$ such that $f(x) = z$. By Lemma 4.4, $|s(x)| = |t(x)|$.

Claim 1. $s' \subseteq n \setminus s(x)$

Proof. Let $i \in s'$. Then, $x(i) \subseteq U_i' \subseteq U_i$. Since U_i is separable, $[x_0(i)] = \{ x_0(i) \}$. So, $s' \notin s(x)$. Thus, $s' \subseteq n \setminus s(x)$. □

So, we get $|s'| \leq n - |s(x)|$.

Claim 2. $n \setminus t' \subseteq t(x)$

Proof. Let $j \in n \setminus t'$. Then, by the definition of z, we have $z(j) \in (a(j), b(j)) \setminus \mathrm{Cl}(L_j \cap M)$. If f_j is constant on $[x]$, then $f(x)(j) = z(j) \in \mathrm{Cl}(L_j \cap M)$. So, f_j is not constant on $[x]$ and hence $j \in t(x)$. $n \setminus t' \subseteq t(x)$. □

So, we get $n - |t'| \leq |t(x)| = |s(x)|$ and so $n - |s(x)| \leq |t'|$. Therefore, we have $|s'| \leq |t'|$.

By Theorem 5.1, f^{-1} is a continuous injection from $\mathrm{ran}(f)$ into $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$ is open. By applying the same argument, we can obtain $|s'| \geq |t'|$. Therefore, $|s'| = |t'|$. □

6. Semigroups and the invariance of domain property

In this section, as an application of Theorem 5.1, we shall show that when the finite product of connected LOTS carries a cancellative topological semigroup, then it is locally Euclidean (Theorem 6.7). First, we shall prove necessary lemmas.

Recall that we say that a topological space X is *locally separable* if and only if every point in X has a separable open neighborhood. We say that X is *locally Euclidean* if and only if there exists a nonnegative integer n such that every point in X has an open neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$. So, every finite product of locally separable connected LOTS without endpoints is locally Euclidean.

Lemma 6.1. *Let X be a connected separable topological space, K a nowhere separable connected LOTS, and $g : X \to K$ a continuous function. Then, g is constant.*

Proof. Let X, K, g be as in the assumption. Since X is connected and separable, so is $\operatorname{ran}(g)$. Suppose that g is not constant. Then, there exists $z_1, z_2 \in \operatorname{ran}(g)$. Without loss of generality, we may assume $z_1 < z_2$. Since $\operatorname{ran}(g)$ is connected, we have $(z_1, z_2) \subseteq \operatorname{ran}(g)$. Then, (z_1, z_2) is a nonempty separable open set in K. This is a contradiction to the assumption that K is nowhere separable. □

Lemma 6.2. *Let K be a nowhere separable connected LOTS and L a separable connected LOTS. Let $g : K \to L$ be a continuous function. Then, g is constant.*

Proof. Let θ be a regular cardinal with $\theta \geq \theta_{K,L,g}$. Let M be a countable elementary substructure of $H(\theta)$ with $K, L, g \in M$. By way of contradiction, suppose that there exist $x_0, x_1 \in K$ such that $g(x_0) \neq g(x_1)$. Without loss of generality, we may assume $x_0 < x_1$. By elementarity, we may assume $x_0, x_1 \in K \cap M$. By reversing the order of L if necessary, we may assume $g(x_0) < g(x_1)$. Let $x_2 = \inf \{ x \in K \mid x_0 \leq x \wedge g(x_0) < g(x) \}$. Then, $x_2 \in M$. By the definition of x_2, $g(\min[x_2]) = g(x_2) = g(x_0)$.

If $x_2 = \max[x_2]$, then $g(\min[x_2]) = g(x_2) = g(\max[x_2])$. By Fact 3.1, g is constant. So, we have $x_2 < \max[x_2]$. By the definition of x_2, there exists $x_3 \in (x_2, \max[x_2])$ such that $g(x_0) < g(x_3)$. Since L is separable, there exists $y_0 \in L \cap M$ such that $g(x_0) < y_0 < g(x_3)$. Then, $g^{\leftarrow}(\leftarrow, y_0)$ is an open neighborhood of x_2 and $g^{\leftarrow}(\leftarrow, y_0) \in M$. So, there exist $a, b \in K \cap M$ such that $x_2 \in (a, b) \subseteq g^{\leftarrow}(\leftarrow, y_0)$. Then, $x_2 < b \leq x_3 < \max[x_2]$. This is a contradiction since $b \in M$ but $(x_2, \max[x_2]) \cap M = \varnothing$. □

By repeatedly applying the previous lemma, the following can be easily shown.

Lemma 6.3. *Let n be a positive integer, K_i a nowhere separable connected LOTS for each $i < n$, and L a separable connected LOTS. Let $g : \prod_{i<n} K_i \to L$ be a continuous function. Then, g is constant.*

We shall temporarily say that a topological space X has the *IOD* property if and only if every continuous injective function $f : X \to X$ is an open map. So, Theorem 5.1 implies that every finite product of connected LOTS without endpoints has the IOD property. We shall use this fact to investigate connected cancellative topological semigroups on the finite product of connected LOTS without endpoints.

When S is a semigroup and $x \in S$, we shall define $l_x : S \to S$ and $r_x : S \to S$ by $l_x(y) = xy$ and $r_x = yx$. Temporarily, we shall write that a topological space X satisfies (+) if and only if for all $x, y \in X$, there exists a homeomorphism f from an open neighborhood of x onto an open neighborhood of y such that $f(x) = y$.

Lemma 6.4. *Let S be a cancellative topological semigroup such that S has the IOD property. Then, S satisfies $(+)$.*

Proof. Let $x, y \in S$. Since S has the IOD property, l_x is an open map. Define $W_x = \operatorname{ran}(l_x)$. Then, W_x is an open neighborhood of $l_x(y) = xy$. Similarly, r_y is an open map. Define $W_y = \operatorname{ran}(r_y)$. Then, W_y is an open neighborhood of $r_y(x) = xy$. Let $W = W_x \cap W_y$. Then, W is an open neighborhood of xy. Let $U_y = (l_x)^{-1}W$ and $U_x = (r_x)^{-1}W$. Then, U_y is an open neighborhood of y while U_x is an open neighobrhood of x. It is easy to see that $l_x \restriction U_y$ and $r_y \restriction U_x$ are homeomorphism onto W. Thus, U_x and U_y are homeomorphic. □

Lemma 6.5. *Let K be a nonempty connected LOTS. Then, there exists a nonempty open subset U of K such that either U is separable or nowhere separable.*

Proof. If K is nowhere separable, any $p \in K$ and $U = K$ works. Suppose that K is not nowhere separable. Then, there exists a nonempty separable open subset U of K. □

Lemma 6.6. *Let n be a positive integer and K_i a connected LOTS with no end points and for each $i < n$. Suppose that $\prod_{i<n} K_i$ satisfies $(+)$. Then, for every $i < n$, K_i is either locally separable or nowhere separable.*

Proof. Let $\vec{K} = \prod_{i<n} K_i$. Suppose otherwise, i.e., there exists $i_0 < n$ such that K_{i_0} is neither locally separable nor nowhere separable.

We shall define $x, x' \in \vec{K}$ as follows. For every $i < n$ with $i \neq i_0$, by Lemma 6.5, there exists a nonempty open subset U_i of K_i such that either U_i is separable or U_i is nowhere separable. Without loss of generality, we may assume that U_i is a convex set. Let $x(i)$ be an arbitrary element of U_i and set $x'(i) = x(i)$. Since K_{i_0} is not nowhere separable, there exists a nonempty convex separable open subset U_{i_0} of K_{i_0}. Let $x(i_0)$ be an arbitrary element of U_{i_0}. Since K_{i_0} is not locally separable, there exists a $x'(i_0) \in K_{i_0}$ that has no separable open neighborhood.

Let s' be the set of all $i < n$ such that U_i is separable. Notice that $i \in s'$ if and only if $x(i)$ has a separable open neighborhood. In particular, since $x(i_0)$ has a separable open neighborhood U_{i_0}, we have $i_0 \in s'$. For every $i < n$ with $i \neq i_0$, since $x'(i) = x(i)$, $i \in s'$ if and only if $x'(i)$ has a separable open neighborhood. By the definition of $x'(i_0)$, $x'(i_0)$ has no separable open neighborhood. Thus, $i \in s' \setminus \{ i_0 \}$ if and only if $x'(i)$ has a separable open neighborhood.

Since $\vec{K}$ satisfies $(+)$, there exists a homeomorphism f from an open neighborhood of x onto an open neighborhood of x' such that $f(x) = x'$. Without loss of generality, we may assume that $\operatorname{dom}(f)$ has the form $\prod_{i<n} K'_i$ where for each $i < n$, K'_i is an open interval in K_i such that

$K_i' \subseteq U_i$. Then, we can consider f as a continuous injection from $\prod_{i<n} K_i'$ into $\prod_{i<n} K_i$. By Lemma 5.2,

$$\begin{aligned}|s'| &= |\{\, i < n \mid x(i) \text{ has a separable open neighborhood}\,\}| \\ &= |\{\, j < n \mid f(x)(j) \text{ has a separable open neighborhood}\,\}| \\ &= \left|\{\, j < n \mid x'(j) \text{ has a separable open neighborhood}\,\}\right| \\ &= |s' \setminus \{\, i_0 \,\}\,|\end{aligned}$$

This is a contradiction. □

Theorem 6.7. *Let n be a positive integer and K_i a connected LOTS with no end points for all $i < n$. If there is a cancellative topological semigroup on $\prod_{i<n} K_i$, then for every $i < n$, K_i is locally separable. In particular, $\prod_{i<n} K_i$ is locally Euclidean.*

Proof. If $|K_i| = 1$, then we may ignore the coordinate. Thus, we may assume that for all $i < n$, $|K_i| \geq 2$.

Let $\vec{K} = \prod_{i<n} K_i$. Suppose that there is a cancellative topological semigroup on $\vec{K}$. By Lemma 6.4, S satisfies (+). By Lemma **??**, for every $i < n$, K_i has no endpoints. By Lemma 6.6, for every $i < n$, K_i is either locally separable or nowhere separable.

Let u_0 be the set of all $i < n$ such that K_i is locally separable and $u_1 = n \setminus u_0$.

Claim 1. For every $j \in u_1$, $x_0, x_0', x_1, x_1' \in \prod_{i\in u_0} K_i$ and $y, y' \in \prod_{i\in u_1} K_i$,

$$\pi_j((x_0 \cup y) \cdot (x_0' \cup y')) = \pi_j((x_1 \cup y) \cdot (x_1' \cup y'))$$

Namely, $\pi_j((x \cup y) \cdot (x' \cup y'))$ does not depend on x, x'.

Proof. Fix j, y, y' as in the assumption. Define a function $\bar{g} : \prod_{i\in u_0} K_i \times \prod_{i\in u_0} K_i \to K_j$ by

$$\bar{g}(x, x') = \pi_j((x \cup y) \cdot (x' \cup y'))$$

Then, since $\bar{g}$ is a continuous function from a separable space into a nowhere separable connected LOTS, by Lemma 6.1, $\bar{g}$ is a constant function. In particular, $\bar{g}(x_0, x_0') = \bar{g}(x_1, x_1')$, which means

$$\pi_j((x_0 \cup y) \cdot (x_0' \cup y')) = \pi_j((x_1 \cup y) \cdot (x_1' \cup y'))$$

□

Define $\rho : \prod_{i<n} K_i \to \prod_{j\in u_1} K_j$ by $\rho(x) = x \restriction u_1$. Let $\sim_\rho$ be an equivalence relation on $\prod_{i<n} K_i$ defined by $\hat{x} \sim_\rho \hat{x}'$ if and only if $\rho(\hat{x}) = \rho(\hat{x}')$, namely $\hat{x} \restriction u_1 = \hat{x}' \restriction u_1$. Then, by the previous claim, for every $\hat{x}_0, \hat{x}_0', \hat{x}_1, \hat{x}_1' \in \prod_{i<n} K_i$, if $\hat{x}_0 \sim_\rho \hat{x}_1$ and $\hat{x}_0' \sim_\rho \hat{x}_1'$, then

$$\rho(\hat{x}_0 \cdot \hat{x}_0') = \rho(\hat{x}_1 \cdot \hat{x}_1')$$

Thus, we can find a binary operation $*$ on $\prod_{i\in u_1} K_i$ so that $(\prod_{i\in u_1} K_i, *)$ is a quotient semigroup of $\prod_{i<n} K_i$ by $\sim_\rho$. Namely, it satisfies that for every $\hat{x}, \hat{x}' \in \prod_{i<n} K_i$,

$$\rho(\hat{x}) * \rho(\hat{x}') = \rho(\hat{x} \cdot \hat{x}')$$

Or equivalently,

$$(\hat{x} \restriction u_1) * (\hat{x}' \restriction u_1) = (\hat{x} \cdot \hat{x}') \restriction u_1$$

Claim 2. For every $j \in u_0$, $x, x' \in \prod_{i\in u_0} K_i$, and $y_0, y_0', y_1, y_1' \in \prod_{i\in u_1} K_i$,

$$\pi_j((x \cup y_0) \cdot (x' \cup y_0')) = \pi_j((x \cup y_1) \cdot (x' \cup y_1'))$$

Proof. Fix j, x, x' as in the assumption. Define a function $\tilde{g} : \prod_{i\in u_1} K_i \to K_j$ by

$$\tilde{g}(y, y') = \pi_j((x \cup y) \cdot (x' \cup y'))$$

Then, since $\tilde{g}$ is a continuous function from the finite product of many nowhere separable connected LOTS into a separable connected LOTS, by Lemma 6.3, $\tilde{g}$ is constant. Thus,

$$\begin{aligned}\pi_j((x \cup y_0) \cdot (x' \cup y_0')) &= \tilde{g}(y_0, y_0') = \tilde{g}(y_1, y_1') \\ &= \pi_j((x \cup y_1) \cdot (x' \cup y_1'))\end{aligned}$$

□

Claim 3. $(\prod_{i\in u_1} K_i, *)$ is cancellative.

Proof. Let $y, y', y'' \in \prod_{i\in u_1} K_i$ such that $y * y'' = y' * y''$. We shall show that $y = y'$. Let $x_0 \in \prod_{i\in u_0} K_i$ and define $\hat{x} = x_0 \cup y$, $\hat{x}' = x_0 \cup y'$, and $\hat{x}'' = x_0 \cup y''$. By the definition of $*$, we have

$$\begin{aligned}(\hat{x} \cdot \hat{x}'') \restriction u_1 &= (\hat{x} \restriction u_1) * (\hat{x}'' \restriction u_1) \\ &= y * y'' = y' * y'' \\ &= (\hat{x}' \restriction u_1) * (\hat{x}'' \restriction u_1) \\ &= (\hat{x}' \cdot \hat{x}'') \restriction u_1\end{aligned}$$

Meanwhile, by Claim 2, since $\hat{x} \restriction u_0 = \hat{x}' \restriction u_0 = \hat{x}''$, we have

$$(\hat{x} \cdot \hat{x}'') \restriction u_0 = (\hat{x}' \cdot \hat{x}'') \restriction u_0$$

Thus, $\hat{x} \cdot \hat{x}'' = \hat{x}' \cdot \hat{x}''$. Since $\vec{K}$ is cancellative, we have $\hat{x} = \hat{x}'$. Thus, $y = y'$. Similarly, we can show that $y * y' = y * y''$ implies $y' = y''$. □

Define $f : \prod_{i\in u_1} K_i \times \prod_{i\in u_1} K_i \to \prod_{i\in u_1} K_i \times \prod_{i\in u_1} K_i$ by

$$f(y, y') = \langle y, y * y' \rangle$$

Claim 4. f is a continuous injection.

Proof. Since $(\prod_{i\in u_1} K_i, *)$ is a topological semigroup, f is continuous. To show that f is injective, suppose $f(y_0, y_0') = f(y_1, y_1')$. Then, $y_0 = y_1$ and $y_0 * y_0' = y_1 * y_1'$. So, we have $y_1 * y_0' = y_1 * y_1'$. Since $*$ is cancellative, we have $y_0' = y_1'$. □

By Theorem 1.3, f is coordinate-wise. Since $f(y, y') = \langle y, y*y'\rangle$, it implies that $y * y'$ depends only on y'. Since $(\prod_{i\in u_1} K_i, *)$ is cancellative, it implies that it is a trivial semigroup. Thus, $u_1 = \varnothing$ and hence $u_0 = n$. Namely, for every $i < n$, K_i is locally separable. So, $\vec{K}$ is locally separable and hence first countable.

□

If S is a monoid in addition, then we may use the following theorem.

Theorem 6.8. *Let S be a connected cancellative topological monoid such that both S and S^2 have the IOD property. Then, S is a group.*

Proof. Let e be the identity element. Define $f : S^2 \to S^2$ by $f(x, y) = \langle x, xy\rangle$. Then, clearly f is continuous.

Claim 1. f is injective.

Proof. Suppose that $f(x, y) = f(x', y')$. Then, $\langle x, xy\rangle = \langle x', x'y'\rangle$. So, $x = x'$ and $xy = x'y'$. Thus, $xy = xy'$. Since S is cancellative, $y = y'$. Thus, $\langle x, y\rangle = \langle x', y'\rangle$. □

Since S^2 has the IOD property, f is an open map.

Claim 2. There exists an open neighborhood U_e of e such that every element of U_e has a right inverse.

Proof. We have $f(e, e) = \langle e, e\rangle$. So, there exists an open neighborhood U_e of e such that $U_e \times U_e \subseteq \mathrm{ran}(f)$. Let $x \in U_e$. Then, $\langle x, e\rangle \in \mathrm{ran}(f)$. So, there exists $y \in S$ such that $f(x, y) = \langle x, e\rangle$. So, $xy = e$. □

Let U be the set of all elements of S that have a right inverse. Trivially, $U_e \subseteq U$. It is easy to see that if $x, y \in U$, then $xy \in U$.

Claim 3. U is open.

Proof. Let $x \in U$. We know that l_x is continuous. Since S is cancellative, l_x is injective. Since S has the IOD property, l_x is an open map. Thus, $l_x{}^{\to}U_e$ is an open neighborhood of x. We shall show that $l_x{}^{\to}U_e \subseteq U$. Let $z \in l_x{}^{\to}U_e$. Then, there exists $y \in U_e$ such that $xy = z$. Since both x and y are in U, we have $z = xy \in U$. □

Claim 4. U is closed.

Proof. Let $x \in S \setminus U$. By the same argument as in the proof of the previous claim, $l_x{}^{\to}U_e$ is an open neighborhood of x. We shall show that $l_x{}^{\to}U_e \subseteq S \setminus U$, i.e. $U \cap l_x{}^{\to}U_e = \varnothing$. Suppose not. Then, there exists $y \in U_e$ such that $xy \in U$. So, xy has a right inverse, say z. Then, $xyz = e$. So, yz is a right inverse of x and hence $x \in U$. This is a contradiction. □

Since S is connected, the previous two claims imply $U = S$. Namely, every element of S has a right inverse. Since S is a cancellative monoid, it implies that every element of S has an inverse. Therefore, S is a group. □

Thus, every cancellative topological monoid on the finite product of connected LOTS is in fact a group. It is well known that the following theorem can be shown by combining the results of A. Gleason [3] and D. Montgomery and L. Zippin [10].

Theorem 6.9 (Gleason, Montgomery, and Zippin). *Every locally Euclidean group is a Lie group.*

Thus, we can obtain the following corollary.

Corollary 6.10. *Every cancellative connected topological monoid on the finite product of connected LOTS is a Lie group.*

Proof. Let S be a cancellative connected topological monoid on the finite product $\vec{K} = \prod_{i<n} K_i$ of connected LOTS. By Theorem 5.1, $\vec{K}$ has the IOD property. By Theorem 6.8, S is a group. By Theorem 6.7, each K_i is locally separable and has no endpoints. Thus, $\vec{K}$ is locally Euclidean. By Theorem 6.9, S is a Lie group. □

Thus, we may use the classification of connected finite-dimensional Lie groups to classify connected cancellative topological monoids on Euclidean spaces. For example, connected two-dimensional Lie groups are classified as follows:

Fact 6.11.
- (i) There are three abelian two-dimensional Lie group up to isomorphism: $\mathbb{R}^2$, $\mathbb{R} \times S^1$, and $S^1 \times S^1$.
- (ii) There exists a unique nonabelian two-dimensional Lie group up to isomorphism, which is the affine group of the line $\mathrm{Aff}(1)$.

Neither $\mathbb{R} \times S^1$ nor $S^1 \times S^1$ is not homeomorphic to $\mathbb{R}^2$. Meanwhile, it is well known that $\mathrm{Aff}(1)$ is homeomorphic to $\mathbb{R}^2$. So, we get the following corollary.

Corollary 6.12.
- *(i) The only abelian cancellative connected topological monoid on the product of two connected LOTS is* $\mathbb{R}^2$ *with the ordinary vector addition up to isomorphism.*
- *(ii) The only nonabelian cancellative connected topological monoid on the product of two connected LOTS is* $\mathrm{Aff}(1)$ *up to isomorphism.*

7. Cancellative topological semigroups on the finite product of connected LOTS possibly with endpoints

Now, we shall consider the case when some K_i has end points. Notice that for every positive integer n, $[0, \infty)^n$ with the coordinate-wise addition is an example of a cancellative topological semigroup on the finite product of connected LOTS that have the minimum element. So, this case is not vacuous.

Lemma 7.1. *Let K be a connected LOTS. Then, there exists a connected LOTS L with no end points such that K is a convex subset of L.*

Proof. If K has no minimum element, then let $K' = K$. Otherwise, let K' be the concatenation of $\mathbb{R}$ and K. It is easy to see that in both cases, K' is a connected LOTS without minimum element and K is a convex subset of K'

If K' has no maximum element, then let $L = K'$. Otherwise, let L be the concatenation of K' and $\mathbb{R}$. Again, in both cases, L is a connected LOTS without maximum or minimum element and K is a convex subset of L. □

Lemma 7.2. *Let n be a positive integer and K_i a connected LOTS. Suppose that there is a cancellative topological semigroup on $\prod_{i<n} K_i$. For each $i < n$, let K_i' be the set of all $p \in K_i$ such that p is neither the maximum or minimum element of K_i. Note that K_i' is a connected LOTS. Then, for all $x, y \in \prod_{i<n} K_i'$, we have $xy \in \prod_{i<n} K_i'$. Thus, the restriction of the topological semigroup on $\prod_{i<n} K_i'$ is a cancellative topological semigroup.*

Proof. By Lemma 7.1, for each $i < n$, there exists a connected LOTS L_i without end points such that K_i is a convex subset of L_i. Let $x_0, y_0 \in \prod_{i<n} K_i'$. Notice that $\prod_{i<n} K_i'$ is an open subset of $\prod_{i<n} K_i$. Let $f : \prod_{i<n} K_i' \to \prod_{i<n} L_i$ be a function defined by $f(y) = x_0 y$. Then, f is clearly a continuous injection. By Theorem 1.6, f is an open map. Thus, $f^{\to} \prod_{i<n} K_i'$ is an open neighborhood of $x_0 y_0$ in $\prod_{i<n} L_i$.

We shall show that $x_0 y_0 \in \prod_{i<n} K_i'$. Suppose not. Then, there exists $j < n$ such that $(x_0 y_0)(j)$ is either the maximum or minimum element of K_j. For simplicity, suppose that $(x_0 y_0)(j)$ is the maximum element of K_j. Since $f^{\to} \prod_{i<n} K_i'$ is an open neighborhood of $x_0 y_0$ in $\prod_{i<n} L_i$, there exists $z \in f^{\to} \prod_{i<n} K_i'$ such that $z(j) > (x_0 y_0)(j)$. However, by the definition, we have $z \in \prod_{i<n} K_i$ and hence $z(j) \in K_j$. Thus, $z(j) \leq (x_0 y_0)(j)$. This is a contradiction. □

Theorem 7.3. *Let n be a positive integer and K_i a connected LOTS for each $i < n$. Suppose that there is a cancellative topological semigroup on $\prod_{i<n} K_i$. Then, for every $i < n$ and $p \in K_i$, if p is neither the maximum or minimum element of K_i, p has a separable open neighborhood in K_i.*

Proof. For each $i < n$, let K_i' be the set of all $p \in K_i$ that is neither the maximum nor minimum element. Then, by Lemma 7.2, there is a cancellative topological semigroup on $\prod_{i<n} K_i'$. By Theorem 6.7, for every $i < n$, K_i' is locally separable. Thus, for every $i < n$ and $p \in K_i'$, p has a separable open neighborhood in K_i'. Since K_i' is an open subspace, p also has a separable open neighborhood in K_i. □

References

1. L. E. J. Brouwer, *Beweis der invarianz des n-dimensionalen gebiets*, Math. Ann. **71** (1911), no. 3, 305–313. MR 1511658
2. ———, *Über Abbildung von Mannigfaltigkeiten*, Math. Ann. **71** (1911), no. 1, 97–115. MR 1511644
3. A. M. Gleason, *Groups without small subgroups*, Ann. of Math. (2) **56** (1952), 193–212. MR 49203
4. T. Ishiu, *Finite products of connected nowhere separable linearly ordered spaces*, Topology Appl. **300** (2021), Paper No. 107763, 21. MR 4291559
5. ———, *Brouwer's fixed-point theorem on the finite product of compact linearly ordered topological spaces*, Topology Appl. **362** (2025), Paper No. 109221, 14. MR 4854756
6. ———, *The finite product of connected linearly ordered sets cannot be embedded into the product of a smaller dimension*, Topology Proceedings **66** (2025).
7. S. Mardešić, *Mapping products of ordered compacta onto products of more factors*, Glasnik Mat. Ser. III **5(25)** (1970), 163–170. MR 275378
8. G. Martínez-Cervantes and G. Plebanek, *The Mardešić conjecture and free products of Boolean algebras*, Proc. Amer. Math. Soc. **147** (2019), no. 4, 1763–1772. MR 3910440
9. C. Miranda, *Un'osservazione su un teorema di Brouwer*, Boll. Un. Mat. Ital. (2) **3** (1940), 5–7. MR 4775
10. D. Montgomery and L. Zippin, *Small subgroups of finite-dimensional groups*, Ann. of Math. (2) **56** (1952), 213–241. MR 49204
11. L. B. Treybig, *Concerning continuous images of compact ordered spaces*, Proc. Amer. Math. Soc. **15** (1964), 866–871. MR 167953