Some results on the products of connected linearly ordered sets

Tetsuya Ishiu

Department of Mathematics
Miami University

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Outline

Coordinate-wise functions

Definition

Let $X_0, \ldots, X_{n-1}, Y_0, \ldots, Y_{m-1}$ be sets. We say that a function $f : \prod_{i<n} X_i \to \prod_{j<m} Y_j$ is *coordinate-wise* if and only if there exists $\pi : m \to n$ such that for every $j < m$, the value of $f(x)(j)$ only depends on $x(\pi(j))$, i.e. there exists $h_j : X_{\pi(j)} \to Y_j$ such that for every $x \in \prod_{i<n} X_i$, $f(x)(j) = h_j(x(\pi(j)))$.

Note that there are so many homeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$ that are not coordinate-wise, for example, $f(x, y) = (x + y, x - y)$.

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The theorem of Eda and Kamijo

Let K be a linearly ordered set and $x \in K$. Then, let $\mathrm{cf}(x)$ and $\mathrm{ci}(x)$ denote the cofinality and coinitiality of x.

Theorem (K. Eda and R. Kamijo)

Let L be a connected linearly ordered set. If either $\mathrm{cf}(x)$ or $\mathrm{ci}(x)$ is uncountable for a dense set of x in L, then every homeomorphism $f : L^n \to L^n$ is coordinate-wise for every natural number n.

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Question (Eda and Kamijo)

What if every nonempty convex set has an Aronszajn suborder?

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Nowhere real linearly ordered sets

Definition

Let K be a linearly ordered set. We say that K is *nowhere real* if and only if no uncountable convex set of K is separable.

In other words, the closure of any countable set is nowhere dense.

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Previous theorem

Theorem (I.)

Let K_0, K_1, L_0, L_1 be nowhere real connected linearly ordered sets. Then, every injective continuous function $f : K_0 \times K_1 \to L_0 \times L_1$ is coordinate-wise.

So, while it is restricted to dimension 2, this theorem improves the result of Eda and Kamijo in the following ways.

- Just being continuous and injective is sufficient.
- These four linearly ordered sets may be different (and they may be the same).
- Now, the completion of an Aronszajn line is covered.

In addition, the proof is totally different.

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Intermediate Value Theorem

Lemma

Let K and L be connected linearly ordered sets, and $g : K \to L$ a continuous function. Let $a < b$ be both in K. If $z \in L$ is between $g(a)$ and $g(b)$, then there exists a $c \in (a, b)$ such that $g(c) = z$.

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1-dimensional notations

Definition

Let K be a connected linearly ordered set, and M a set (typically a countable elementary submodel of $H(\theta)$).
Let $J(K, M)$ be the set of all $x \in K \setminus cl(K \cap M)$ with $\inf(K \cap M) \leq x \leq \sup(K \cap M)$.
For every $x \in K$, let

$$\eta(x) = \eta(K, M, x) = \sup\{\, y \in \mathrm{cl}(K \cap M) : y \leq x \,\}$$
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$$I(x) = I(K, M, x) = [\eta(K, M, x), \zeta(K, M, x)]$$
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if they exist. When it is clear from the context, we may omit K and M.

Note that if $x \in J(K, M)$, then all of them exist.

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Let $\vec{K} = \prod_{i<n} K_i$ where each K_i is a connected linearly ordered sets.

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Setting

Now suppose $K_0, \ldots, K_{n-1}, L$ are nowhere real connected linearly ordered sets. Let $\vec{K} = \prod_{i<n} K_i$. We shall consider a continuous function $g : \vec{K} \to L$.

Let M be a countable elementary submodel of $H(\theta)$ with $K_0, \ldots, K_{n-1}, L, g \in M$.

We will assume that all linearly ordered sets are pairwise disjoint.

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Max and min at corners

Lemma

Let $x \in \vec{K}$ be so that for every $i < n$, $\inf(K_i \cap M) \le x(i) \le \sup(K_i \cap M)$. Then,

$$\max g^{\rightarrow}(\vec{I}(x)) = \max g^{\rightarrow}(\vec{C}(x))$$

$$\min g^{\rightarrow}(\vec{I}(x)) = \min g^{\rightarrow}(\vec{C}(x))$$

In particular, if g is constant on $\vec{C}(x)$, then g is constant on $\vec{I}(x)$.

Note that for example, if $x(i) \in J(K_i, M)$ for every $i < k$, and $x(i) \in \mathrm{cl}(K_i \cap M)$ for every $k \le i < n$, then

$$\vec{I}(x) = \prod_{i<k} I(x(i)) \times \prod_{k \le i < n} \{x(i)\}$$

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Moving inside $I(g(x))$

By using the previous lemma and its proof, we can show the following lemma.

Lemma

Let $x \in \vec{K}$ be so that for every $i < n$, $\inf(K_i \cap M) \leq x(i) \leq \sup(K_i \cap M)$.

1. *If $g(x) \in M$, then g is constant on $\vec{I}(x)$.*
2. *If $g(x) \in J(L, M)$, then*
   - $g^{\rightarrow}(\vec{I}(x)) = I(g(x))$, *and*
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Now, let $K_0, \ldots, K_{n-1}, L_0, \ldots, L_{n-1}$ be nowhere real connected linearly ordered sets. Let $\vec{K} = \prod_{i<n} K_i$ and $\vec{L} = \prod_{j<n} L_j$.

Let M be a countable elementary submodel of $H(\theta)$ with $K_0, \ldots, K_{n-1}, L_0, \ldots, L_{n-1} \in M$.

Let $f : \vec{K} \to \vec{L}$ is an injective continuous function. For every $j < n$, let $g_j : \vec{K} \to L_j$ be given by $g_j(x) = f(x)(j)$ (i.e. the j-th component function).

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We go by induction on n, so suppose that the theorem holds for $<n$-dimensional cases.

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Lemma

Let $\hat{x} \in \vec{K}$ be so that $\hat{x}(0)$ is a limit point of $K_0 \cap M$, and for every $1 \leq i < n$, $\hat{x}(i) \in J(K_i, M)$.
Then, there exists a unique $\hat{j} < n$ such that for every $x \in \vec{I}(\hat{x})$, $g_{\hat{j}}$ is constant on $\vec{I}(\hat{x})$.

To prove this, take $p \in K_0$ that is sufficiently close to $\hat{x}(0)$, and for every $c \in \vec{C}(\hat{x})$, compare $f(c)$ and $f(p^\frown c \restriction [1, n))$. We can show that for every $j < n$, if g_j is not constant on $\vec{I}(\hat{x})$, then $g_j(c) = g_j(p^\frown c \restriction [1, n))$. Since f is injective, by using the inductive hypothesis, we can find a unique $\hat{j}$.

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Global Lemma

By pasting this local result, we can prove the following lemma.

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Let p be a limit point of $K_0 \cap M$. Then, there exists $\hat{j} < n$ such that $g_{\hat{j}}$ is constant on $\{ x \in \vec{J}(\vec{K}, M) \mid x(0) = p \}$.

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Outline

No analogue of Peano curve

As a byproduct of this research, we showed the following theorem, too.

Theorem

Let K, L_0, L_1 be nowhere real connected linearly ordered sets such that K has the minimum and maximum elements. Then, there is no continuous surjective function from K to $L_0 \times L_1$.

So, there cannot be an analogue of Peano curve in this situation.
Note that if K has the maximum and minimum elements, then K is compact.

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Proof

Let $f : K \to L_0 \times L_1$ be a continuous surjective function. Let M be a countable elementary substructure of $H(\theta)$ with $K, L_0, L_1, f \in M$ for a sufficiently large regular cardinal θ. Let $g_0 : K \to L_0$ and $g_1 : K \to L_1$ be the component functions of f.

Let $\langle y_i : i < \omega \rangle$ be an increasing sequence in $L_0 \cap M$ and $z \in J(L_1, M)$. Then, for every $i < \omega$, there exists an $x_i \in K$ such that $f(x_i) = \langle y_i, z \rangle$.

Since $z \in J(L_1, M)$, we have $x_i \notin \mathrm{cl}(K \cap M)$ for every $i < \omega$. Since K is compact, we have $x_i \in J(K, M)$. Then, g_0 is constant on $I(x_i)$, i.e. for every $x \in I(x_i)$, $g_0(x) = y_i$.

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Hence, for every $i < \omega$, $\{ g_1(\eta(x_i)), g_1(\zeta(x_i)) \} = C(z)$. Let $a_i, b_i \in C(x_i)$ be so that $g_1(a_i) = \eta(z)$ and $g_1(b_i) = \zeta(z)$.

Since K is compact, there exists a limit point x_ω of $\{ x_i \mid i < \omega \}$. Then, clearly x_ω is limit points of both $\{ a_i \mid i < \omega \}$ and $\{ b_i \mid i < \omega \}$. Since x_ω is a limit point of $\{ a_i \mid i < \omega \}$, $g_1(x_\omega) = \eta(z)$. On the other hand, since x_ω is a limit point of $\{ b_i \mid i < \omega \}$, $g_1(x_\omega) = \zeta(z)$. This is a contradiction.

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Open problems

- Can we weaken “linearly ordered”?
- Can there be an analogue of Peano curve if K has no maximum or minimum elements?
- How about other things that can be done for $\mathbb{R}$?