# A proof of Čertanov's Theorem by using countable elementary submodels

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# Space-filling curves

A famous theorem of G. Peano in 1890 says that there exists a continuous surjection from $[0, 1]$ onto $[0, 1] \times [0, 1]$.

It was a groundbreaking result, which challenged the notion of dimensions.

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For which linearly order topological spaces (LOTS) $L$, is there exists a continuous surjection from $L$ onto $L \times L$.

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In 1960, S. Mardešić and P. Papić proved the following result, strengthening Kurepa's result.

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*If $X$ and $Y$ are nondegenerate connected compact Hausdorff spaces and $X \times Y$ is a continuous image of a connected compact LOTS, then both $X$ and $Y$ are metrizable.*

Now, $X$ and $Y$ are not necessarily LOTS.

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Several proofs are given to this theorem.
Since $X$ and $Y$ are compact and metrizable, both of them are separable.

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**Theorem**

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# The use of countable elementary submodels

I proved the following theorem a few years ago.

**Theorem (Ishiu)**

*Let $n$ be a non-zero natural number, $K_0, \ldots, K_{n-1}, L_0, \ldots, L_{n-1}$ be connected nowhere separable LOTS, and $f : \prod_{i<n} K_i \to \prod_{i<n} L_i$ a continuous injection. Then, $f$ is coordinate-wise.*

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I wondered if this technique can be used to prove other theorems and found that it can be used to give another proof of Čertanov's Theorem.

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# $\eta, \zeta, I$

We shall outline the proof. Let $K$ be a countably compact GO-space and $M$ a countable elementary submodel of $H(\theta)$ with $K \in M$ for some sufficiently large regular cardinal $\theta$. We shall defined the following: for every $p \in K$,

$$\eta(K, M, p) = \sup \{ x \in \mathrm{cl}(K \cap M) \mid x \leq p \}$$
$$\zeta(K, M, p) = \inf \{ x \in \mathrm{cl}(K \cap M) \mid x \geq p \}$$
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# $\eta, \zeta, I$ (Cont.)

We shall use the following easy lemma.

**Lemma**

*Let $p, q \in K$ with $p < q$. If $I(p) \neq I(q)$, then there exists $x \in K \cap M$ such that $p \leq x \leq q$.*

**Proof.**

Suppose that there is no $x \in K \cap M$ with $p \leq x \leq q$. Then, we can show that $\eta(p) = \eta(q)$ and $\zeta(p) = \zeta(q)$. So, $I(p) = I(q)$. □

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Let $X$ be a nonseparable Hausdorff space and $g : K \to X$ a continuous function with $g \in M$.

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*If $g(p) \notin \mathrm{cl}(X \cap M)$, then $p \notin \mathrm{cl}(K \cap M)$.*

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# $g : K \to X$ (Cont.)

We can prove the following lemma.

## Lemma

*If $p \in K \setminus \mathrm{cl}(K \cap M)$ and $g(p) \in M$, then either $g(p) = g(\eta(p))$ or $g(p) = g(\zeta(p))$.*

*In particular, $|(g^{\to} I(p)) \cap M| \leq 2$.*

## Idea of the proof.

We shall give a proof when neither $\eta(p)$ nor $\zeta(p)$ belongs to $M$. Note that for all $a, b \in K \cap M$ with $a < \eta(p)$ and $b > \zeta(p)$, by elementarity, there exists $x \in (a, b) \cap M$ such that $g(x) = g(p)$. By the definition of $\eta(p)$ and $\zeta(p)$, either $a < x \leq \eta(p)$ or $\zeta(p) \leq x < b$. By making $a \to \eta(p)$ and $b \to \zeta(p)$, we get the conclusion. □

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Now, we shall give a proof of Čertanov's Theorem. First, we shall prove the following lemma, which is an easy corollary of Čertanov's Theorem.

## Lemma

*Let $K$ be a countably compact GO-space, $X$ and $Y$ infinite compact Hausdorff spaces and $f : K \to X \times Y$ a continuous surjection. Then, both $X$ and $Y$ are separable.*

## Proof.

Let $M$ be a countable elementary submodel of $H(\theta)$ that knows everything in this context for some sufficiently large regular cardinal $\theta$. Suppose that $X$ is not separable. Then, there exists $x_0 \in X \setminus \mathrm{cl}(X \cap M)$. □

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For each $n < \omega$, since $x_0 \notin \mathrm{cl}(X \cap M)$, $t_n \in K \setminus \mathrm{cl}(K \cap M)$ and $g_2(t_n) = y_n$.
For every $n < \omega$, we have $y_{2n} < y_{2n+1} < y_{2n+2}$ and hence $g_2(t_{2n}) < g_2(t_{2n+1}) < g_2(t_{2n+2})$. Since $|(g_2^{\rightarrow} I(t_n)) \cap M| \leq 2$, we have $I(t_{2n}) \neq I(t_{2n+2})$. So, for each $n < \omega$, there exists $u_n \in K \cap M$ such that $t_{2n} < u_n < t_{2n+2}$. $\square$

# The proof of Čertanov's Theorem (Cont.)

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Let $g_1, g_2$ be the coordinate functions of $f$.
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# The proof of Čertanov's Theorem (Cont.)

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Since $K$ is countably compact, there exists the supremum $t_\omega$ of $\{ t_{2n} \mid n < \omega \}$. Since $g_1(t_{2n}) = x_0$, we have $g_1(t_\omega) = x_0 \notin \mathrm{cl}(X \cap M)$

Meanwhile, notice that $t_\omega$ is also the supremum of $\{ u_n \mid n < \omega \}$. For each $n < \omega$, since $u_n \in M$, we have $g_1(u_n) \in M$. Thus, $g_1(t_\omega) \in \mathrm{cl}(X \cap M)$.

This is a contradiction. $\square$

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# Finishing it up

By using this lemma, we may use Treybig's Theorem to finish proving Čertanov's Theorem.

**Proof of Čertanov's Theorem.**

By the lemma, $X$ and $Y$ are separable. So, there exists a closed separable subspace $K'$ of $K$ such that $f^{\rightarrow}K' = X \times Y$. Notice that $K'$ is a compact LOTS. By Treybig's Theorem, both $X$ and $Y$ are metrizable. $\square$

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# The Mardešić Conjecture

So, there is no new theorem. But let me explain one problem in which this argument may help.

G. Martínez-Cervantes and G.Plebanek showed the following theorem, which solved the Mardešić Conjecture proposed in 1970.

**Theorem**

*Let $d$ and $s$ be positive integers. Let $L_1, L_2, \ldots, L_d$ be compact LOTS and $K_1, K_2, \ldots, K_{d+s}$ infinite Hausdorff spaces. If there exists a continuous surjection from $L_1 \times L_2 \times \cdots \times L_d$ onto $K_1 \times K_2 \times \cdots K_{d+s}$, then there exist at least $s+1$-many metrizable factors $K_j$.*

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We may wonder if the Mardešić Conjecture holds for the product of countably compact GO-spaces. Namely

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