The Mardešić Conjecture for countably compact spaces

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Space-filling curves

A famous theorem of G. Peano in 1890 says that there exists a continuous surjection from $[0, 1]$ onto $[0, 1] \times [0, 1]$.

It was a groundbreaking result, which challenged the notion of dimensions.

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For which linearly order topological spaces (LOTS) L, does there exist a continuous surjection from L onto $L \times L$?

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For every nondegenerate connected compact Suslin line S, there is no continuous surjection from S onto $S \times S$.

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In 1964, L. B. Treybig proved the following result.

Theorem (L. B. Treybig)

If X and Y are infinite Hausdorff spaces and $X \times Y$ is a continuous image of a compact LOTS, then both X and Y are metrizable.

Since X and Y are compact and metrizable, both of them are separable.

Corollary

If L is a compact LOTS that is not metrizable, then there exists no continuous surjection from L to $L \times L$.

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The Mardešić Conjecture

S. Mardešić proposed the following conjecture in 1970, which was proved by G. Martínez-Cervantes and G. Plebanek in 2019.

Theorem (G. Martínez-Cervantes and G. Plebanek)

Let d and s be positive integers. Let K_i be a compact LOTS for each $i < d$, X a compact subspace of $\prod_{i<d} K_i$, Z_j an infinite Hausdorff space for each $j < d+s$. If there exists a continuous surjection from X onto $\prod_{j<d+s} Z_j$, then there exist at least $s+1$-many metrizable factors Z_j.

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What does it mean?

If $d = s = 1$, then it coincides with Treybig's Theorem.
By using Peano's Theorem, it is easy to see that for all positive integers d and s, there exists a continuous surjection from $[0, 1]^d$ onto $[0, 1]^{d+s}$. The Mardešić Conjecture implies even this seemingly weaker phenomenon does not happen to a nonmetrizable compact LOTS.

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Countably compact version

Recall that Čertanov's Theorem is the countably compact version of Treybig's Theorem. As the Mardešić Conjecture is a generalization of Treybig's Theorem, we may wonder if we can prove the countably compact version of the Mardešić Conjecture. We solved this problem positively, namely proved the following theorem.

Theorem (T. Ishiu)

Let d and s be positive integers. Let K_i be a compact LOTS for each $i < d$, X a countably compact subspace of $\prod_{i<d} K_i$, Z_j an infinite Hausdorff space for each $j < d + s$. If there exists a continuous surjection f from X onto $\prod_{j<d+s} Z_j$, then there exist at least $s + 1$-many compact and metrizable factors Z_j.

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S. Purish proved that Stone-Čech compactification of any countably compact GO-space is a LOTS. By using this result, we can obtain the following corollary.

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Let me outline the proof of the main theorem.

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Main Lemma

The following lemma is the most significant piece of the proof of the main theorem.

Lemma

Let d, K_i, X, Z_j, and f be as in the assumption of the main theorem. Suppose that Z_d is not separable. Then, there exists a compact LOTS $\tilde{K}_i$ for each $i < d - 1$, a countably compact subspace $\tilde{X}$ of $\prod_{i<d-1} \tilde{K}_i$, and a continuous surjection $\tilde{f} : \tilde{X} \to \prod_{j<d} Z_j$.

Namely, if one of Z_j's is nonseparable, then there exist $\tilde{K}_i$, $\tilde{X}$, and $\tilde{f}$ that satisfies the assumption of the main theorem for $d - 1$. This lemma is proved by the heavy use of countable elementary submodels.

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I(K, M, p)

We shall sketch the proof of the main lemma.
In general, we shall define the following notations.

Definition

Let K be a compact LOTS and M a countable set. Then, for all $p \in K$, define

$$\eta(K, M, p) = \sup \{ u \in \mathrm{cl}(K \cap M) \mid u \leq p \}$$
$$\zeta(K, M, p) = \inf \{ u \in \mathrm{cl}(K \cap M) \mid u \geq p \}$$
$$I(K, M, p) = [\eta(L, M, p), \zeta(L, M, p)]$$

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Set up

Let d, K_i, X, Z_j, and f be as in the assumption of the main lemma.

- Let f_d be the d-th component function of f.
- Let $g : X \to \prod_{j<d} Z_j$ be defined by $g(x) = f(x) \restriction d$. (Technically, g must be a slight extension of this, but never mind.)
- Let M be a good countable elementary submodel of $H(\theta)$ for a sufficiently large regular cardinal θ.

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The First Claim

First, we can prove the following claim.

Claim

Let $r \in Z_d \setminus \mathrm{Cl}(Z_d \cap M)$. Then, there exist finite sequences $\langle i_n | n < \hat{n} \rangle$ and $\langle p_n | n < \hat{n} \rangle$ such that for all $n < \hat{n}$, $i_n < d$ and $p_n \in K_{i_n}$, and

$$f_d^{\leftarrow}\{r\} \subseteq \bigcup_{n<\hat{n}} \pi_{i_n}^{\leftarrow} I(K_{i_n}, M, p_n)$$

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The Second Claim

We can remove the reference to r, which does not belong to M, by proving the following claim.

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There exist finite sequences $\langle i_n | n < \hat{n} \rangle$ and $\langle p_n | n < \hat{n} \rangle$ such that for all $n < \hat{n}$, $i_n < d$ and $p_n \in K_{i_n}$, and

$$\prod_{j<d} Z_j = g^{\rightarrow} \left(X \cap \bigcup_{n<\hat{n}} \pi_{i_n}{}^{\leftarrow} I(K_{i_n}, M, p_n) \right)$$

$I(K_{i_n}, M, p_n)$ can be 'approximated' by elements of M. So, we can almost describe this phenomenon within M.

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An argument with elementarity proves the following third claim.

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By 'combining' $\pi_{i_n}{}^{\leftarrow}\{\,\eta(K_{i_n}, M, p_n), \zeta(K_{i_n}, M, p_n)\,\}$ for all $n < \hat{n}$, we can finish proving the main lemma.

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Toward separability

Let d, K_i, X, Z_j, and f be as in the assumption of the main theorem.

Let $e = |\{ j < d+1 \mid Z_j \text{ is separable} \}|$. Without loss of generality, we may assume that for all $j < d+1$, Z_j is separable if and only if $j < e$.

By repeatedly applying the main lemma, we can prove the following:

- $e \geq 2$, and
- there exists a compact LOTS $\tilde{K}_i$ for each $i < e-1$, a countably compact subspace $\tilde{X}$ of $\prod_{i<e-1} \tilde{K}_i$, and a continuous surjection $\tilde{f} : \tilde{X} \to \prod_{j<e} Z_j$.

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Let $e = |\{ j < d+1 \mid Z_j \text{ is separable} \}|$. Without loss of generality, we may assume that for all $j < d+1$, Z_j is separable if and only if $j < e$.

By repeatedly applying the main lemma, we can prove the following:

- $e \geq 2$, and
- there exists a compact LOTS $\tilde{K}_i$ for each $i < e-1$, a countably compact subspace $\tilde{X}$ of $\prod_{i<e-1} \tilde{K}_i$, and a continuous surjection $\tilde{f} : \tilde{X} \to \prod_{j<e} Z_j$.

Note that $\prod_{j<e} Z_j$ is separable.

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Compactness

By using the fact that $\prod_{j<e} Z_j$ is separable, there exist a separable compact LOTS K_i' for each $i < e-1$, X' a compact subspace of $\prod_{i<e-1} K_i'$, and a continuous surjection $f' : X' \to \prod_{j<e} Z_j$.

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Mardešić's Theorem

In fact, S. Mardešić proved that the Mardešić Conjecture holds when all Z_j's are separable. So, we can finish proving the main theorem without using the full Mardešić Conjecture.
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