A proof of the Mardešić Conjecture by using countable elementary submodels.

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Space-filling curves

Recall the following famous theorem proved by G. Peano in 1890.

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There exists a continuous surjection from $[0,1]$ *onto* $[0,1] \times [0,1]$.

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This situation is very different when we consider nonseparable compact linearly ordered topological spaces (LOTS).

In 1964, L. B. Treybig proved the following theorem called *Treybig Product Theorem*.

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If X and Y are infinite compact Hausdorff spaces and $X \times Y$ is a continuous image of a compact LOTS, then both X and Y are metrizable.

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So, intuitively, we cannot build a continuous surjection from a 'one-dimensional space'(i.e. LOTS) onto a 'space of dimension $\geq$2' in case of nonseparable spaces.

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The Mardešić Conjecture

In 2018, G. Martínez-Cervantes and G.Plebanek showed the following theorem. This statement is called the Mardešić Conjecture because it was proposed by him in 1970.

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Let d and s be positive integers. Let K_i be a compact LOTS for each $i < d$ and Z_j an infinite Hausdorff space for each $j < d + s$. If there exists a continuous surjection from $\prod_{i<d} K_i$ onto $\prod_{j<d+s} Z_j$, then there exist at least $(s+1)$-many metrizable factors Z_j.

Their argument uses the *free dimension*, which is introduced in the same paper. It plays the role of the intuitive dimension in the previous slide.

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Countably compact version of the Mardešić Conjecture

Given Čertanov's Theorem, it is natural to ask if we may replace 'compact LOTS' by 'countably compact GO-space' in the Mardešić Conjecture. Namely,

Question

Let d and s be positive integers. Let K_i be a countably compact GO-space for each $i < d$ and Z_j an infinite Hausdorff spaces for each $j < d + s$. If there exists a continuous surjection from $\prod_{i<d} K_i$ onto $\prod_{j<d+s} Z_j$, then do there exist at least $(s + 1)$-many metrizable factors Z_j?

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Countably compact version (Cont.)

The proof of G. Martínez-Cervantes and G. Plebanek seems heavily dependent on the compactness and cannot be modified to answer this question.

I will present an outline of the proof that gives a positive answer to this question. It is done by using countable elementary submodels.

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Let K be a countably compact GO-space and M a countable elementary submodel of $H(\theta)$ with $K \in M$ for some sufficiently large regular cardinal θ. We shall defined the following: for every $p \in K$,

$$\eta(K, M, p) = \sup \{ q \in \mathrm{cl}(K \cap M) \mid q \leq p \}$$
$$\zeta(K, M, p) = \inf \{ q \in \mathrm{cl}(K \cap M) \mid q \geq p \}$$
$$I(K, M, p) = [\eta(K, M, p), \zeta(K, M, p)]$$

Let me ignore the case $p < \inf(K \cap M)$ or $p > \sup(K \cap M)$.

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Let's begin

Now, we shall start the proof.
We shall prove the following slightly stronger statement.

Theorem

Let d, s be positive integers. For each $i < d$, K_i is a countably compact GO-space, and for each $j < d+s$, Z_j is an infinite Hausdorff space. Suppose that there exist a countably compact subspace Y of $\prod_{i<d} K_i$ and a continuous surjection $f : Y \to \prod_{j<d+s} Z_j$. Then, there exist at least $(s+1)$-many factors Z_j such that Z_j is metrizable.

Namely, f can be a partial function with countably compact domain.

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By using the observation by G. Martínez-Certvantez and G. Plebanek, we may focus on the case of $s = 1$.

We shall show the following lemma.

Lemma

Let $K_i (i < d)$, $Z_j (j < d + 1)$, Y, and f be as in the assumption of the theorem. Suppose Z_0 is not separable. Then, there exist countably compact GO-spaces $K'_i (i < d - 1)$, a countably compact subspace Y' of $\prod_{i<d-1} K'_i$, and a continuous surjection $g : Y' \to \prod_{1 \leq j < d+1} Z_j$.

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By repeatedly applying this lemma, we can prove the following.

Lemma

Let $K_i (i < d)$, $Z_j (j < d + 1)$, Y, and f be as in the assumption of the theorem. Suppose that for all $j < n$, Z_j is not separable. Then, there exist countably compact GO-spaces $K'_i (i < d - n)$, a countably compact subspace Y' of $\prod_{i<d-n} K'_i$, and a continuous surjection $g : Y' \to \prod_{n \le j < d+1} Z_j$.

If there are $(d - 1)$-many nonseparable Z_j's, then we can get to the situation where we can apply Čertanov's Theorem. Otherwise, we can get to the situation where we can apply Mardešić's Theorem.

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The proof of the lemma

Now, the proof of our theorem is reduced to the following lemma.

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Let M be a countable elementary submodel of $H(\theta)$ for a sufficiently large regular cardinal θ such that M knows everything in this context.

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Stars

Since Z_0 is not separable, there exists $z_0 \in Z_0$ such that $z_0 \notin \mathrm{cl}(Z_0 \cap M)$. Let f_j be the coordinate function of f.

Claim

There exist finitely many elements $x_0, x_1, \ldots, x_{m-1} \in Y$ such that for all $x \in Y$, if $f_0(x) = z_0$, then there exists $i < d$ and $k < m$ such that $x(i) \in I(K_i, M, x_k(i))$.

Here $x(i)$ denotes the i-th component of x. For each $i < d$, let π_i be the projection onto K_i, i.e., $\pi_i(x) = x(i)$. Then, $x(i) \in I(K_i, M, x_k(i))$ can be written as $x \in {\pi_i}^{\leftarrow} I(K_i, M, x_k(i))$.

Proof

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Suppose not. Then, there exists a sequence $\langle x_k | k < \omega \rangle$ in Y such that for all $k' < k < \omega$ and $i < d$, $f_0(x_k) = z_0$ and $x_k(i) \notin I(K_i, M, x_{k'}(i))$. Without loss of generality, for every $i < d$, $\langle x_k(i) | k < \omega \rangle$ is monotone. Let x be the limit point of $\{ x_k \mid k < \omega \}$. Then, clearly $f_0(x) = z_0$.
However, by using the assumption that for every $k < \omega$ and $i < d$, $x_{k+1}(i) \notin I(K_i, M, x_k(i))$, we can build a sequence $\langle x'_k | k < \omega \rangle$ in $Y \cap M$ such that $x \in \mathrm{cl}\,\{ x'_k \mid k < \omega \}$ and hence $f_0(x) \in \mathrm{cl}(Z_0 \cap M)$. This is a contradiction. □

Boards

By looking at each coordinate of x_k's, we can show the following claim.

Claim

There exist a finite set $\{\langle i_k, p_k\rangle \mid k < m\}$ such that for all $k < m$, $i_k < d$, $p_k \in K_{i_k}$, and for all $x \in Y$, if $f_0(x) = z_0$, then there exists $k < m$ such that $x \in \pi_{i_k}{}^{\leftarrow} I(K_{i_k}, M, p_k)$.

Remark

By the previous claim, since f is surjective, for every $z \in \prod_{0<j<d+1} Z_j$, there exists $x \in Y \cap \bigcup_{k<m} \pi_{i_k}{}^{\leftarrow} I(K_{i_k}, M, p_k)$ such that $f_{\neq 0}(x) = z$. Here $f_{\neq 0}(x) = f(x) \restriction [1, d+1)$.

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Hyperplanes

For each $k < m$, we shall replace a 'board' $\pi_{i_k}{}^{\leftarrow} I(K_{i_k}, M, p_k)$ by finitely many countably compact subspaces of the products of $(d-1)$-many countably compact GO-spaces.

Two of them are $\pi_{i_k}{}^{\leftarrow}\{\eta(K_{i_k}, M, p_k)\}$ and $\pi_{i_k}{}^{\leftarrow}\{\zeta(K_{i_k}, M, p_k)\}$.

There can be one element $p'_k \in M$ of the Dedekind completion of K_{i_k} that belongs to $I(K_{i_k}, M, p_k)$. We can consider the 'limits' of f from the above and below p'_k. These limits may not be taken in $\prod_{i<d, i \neq i_k} K_i$, so we may need to extend each K_i to a larger GO-space. But we can do it.

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The end of the proof of the lemma

By combining all hyperplanes and associated functions made in the previous slide, we can find countably compact GO-spaces $K'_{k,i}(k < m', i < d-1)$, countably compact subspaces $Y_k(k < m')$ of $\prod_{i<d-1} K'_{k,i}$, and continuous functions $g_k : Y_k \to \prod_{0<j<d+1} X_j$ such that $\bigcup_{k<m'} \mathrm{ran}(g_k) = \prod_{0<j<d+1} X_j$.
We can combine them to construct countably compact GO-space $K'_k(k < d-1)$, a countably compact subspace Y' of $\prod_{i<d-1} K'_k$, and continuous surjection $g : Y' \to \prod_{0<j<d+1} X_j$.
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